"Applied Optimal Control: Optimization, Estimation, and Control"

by **Arthur E. Bryson, Jr.** and **Yu-Chi Ho** (Bryson & Ho, 1969/1975).

- **Chapter 1: Parameter Optimization Problems** Introduces foundational optimization concepts without dynamic variables, focusing on static systems with and without equality/inequality constraints, Lagrange multipliers, and basic numerical gradients.
- **Chapter 2: Optimization Problems for Dynamic Systems** Expands these optimization techniques into time-varying systems, dealing with calculus of variations, Euler-Lagrange equations, and systems with fixed or unspecified terminal times.
- **Chapter 3: Optimization Problems for Dynamic Systems with Path Constraints**
- **Chapter 4: Optimal Feedback Control** Shifts the perspective from open-loop trajectories to closed-loop feedback control laws. It covers key paradigms such as the extremal field approach, Hamilton-Jacobi-Bellman (HJB) equations, and dynamic programming.
- **Chapter 5: Linear Systems with Quadratic Criteria: Linear Feedback** This chapter introduces the Linear Quadratic Regulator (LQR) problem, bridging the gap between optimal control theory and practical linear feedback design by minimizing a quadratic performance index subject to linear differential equation constraints.

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CHAPTER 1

PARAMETER OPTIMIZATION PROBLEMS

PROBLEM 1.3.1

Minimize $L = x^2+y^2+z^2$ (1)

Subject to $f_1 \equiv x+2y+3z-10 = 0$ (2)

$f_2 \equiv x-y+2z-1 = 0$ (3)

Form $H = L+\lambda_1 f_1 + \lambda_2 f_2$

Necessary conditions for a stationary value of H are,

$$\frac{\partial H}{\partial x} = 2x+\lambda_1+\lambda_2 = 0 \quad (4)$$

$$\frac{\partial H}{\partial y} = 2y+2\lambda_1+(-\lambda_2) = 0 \quad (5)$$

$$\frac{\partial H}{\partial z} = 2z+3\lambda_1+2\lambda_2 = 0 \quad (6)$$

$$\frac{\partial H}{\partial \lambda_1} = x+2y+3z-10 = 0 \quad (2)$$

$$\frac{\partial H}{\partial \lambda_2} = x-y+2z-1 = 0 \quad (3)$$

These five necessary conditions may be solved for the five unknowns x, y, z, λ_1 and λ_2.

The point nearest origin satisfying given conditions is given by $x = \frac{19}{59}$, $y = \frac{146}{59}$ and $z = \frac{93}{59}$.

PROBLEM 1.3.2

Maximize $P = 4(x+y)$ (1)

Subject to $f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ (2)

Form $H \equiv P + \lambda f$ (3)

Necessary conditions for stationarity are,

$$\frac{\partial H}{\partial x} = 4 + \frac{2\lambda x}{a^2} = 0 \quad (4)$$

$$\frac{\partial H}{\partial y} = 4 + \frac{2\lambda y}{b^2} = 0 \quad (5)$$

$$\frac{\partial H}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad (2)$$

Solving these three equations we get,

$$x = \frac{\pm a^2}{\sqrt{a^2+b^2}}, \quad y = \frac{\pm b^2}{\sqrt{a^2+b^2}}$$

PROBLEM 1.3.3

Maximize $V = 8xyz$ (1)

Subject to $f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ (2)

Form $H \equiv V + \lambda f$ (3)

Necessary conditions for stationarity are,

$$\frac{\partial H}{\partial x} = 8yz + \frac{2\lambda x}{a^2} = 0 \quad (4)$$

$$\frac{\partial H}{\partial y} = 8xz + \frac{2\lambda y}{b^2} = 0 \quad (5)$$

$$\frac{\partial H}{\partial z} = 8xy + \frac{2\lambda z}{c^2} = 0 \quad (6)$$

$$\frac{\partial H}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad (2)$$

Solving the above four equations, we get.

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}.$$

Hence a rectangular parallelopiped of maximum volume that can be contained in a given ellipsoid [equation (2)] is given by,

$$x = \pm \frac{a}{\sqrt{3}} \quad , \quad y = \pm \frac{b}{\sqrt{3}} \quad \text{and} \quad z = \pm \frac{c}{\sqrt{3}} \; .$$

PROBLEM 1.3.4

Maximize $$L = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u \qquad (1)$$

Subject to $$f(x,u) \equiv x + Gu + c = 0 \qquad (2)$$

Note that (2) is a vector constraint. Also, $L \geq 0$ and Q and R can be taken to be symmetric matrices, without loss of generality. Let $Q = Q_a + Q_s$ such that $Q_a^T = -Q_a$ and $Q_s^T = Q_s$. Then,

$x^T Q x = x^T Q_s x + x^T Q_a x$. But $x^T Q_a x$ is a scalar and hence is equal to its transpose $x^T Q_a^T x$ Thus,

$$x^T Q_a x = x^T Q_a^T x = x^T (-Q_a) x = 0$$

so that,

$$x^T Q x = x^T Q_s x \; .$$

This shows that Q could be taken as a symmetric matrix without any loss of generality.

Form $$H = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (x + Gu + c) \quad (3)$$

Necessary conditions for a minimum are,

$$\frac{\partial H}{\partial x} = x^T Q + \lambda^T = 0 \quad (4)$$

$$\frac{\partial H}{\partial u} = u^T R + \lambda^T G = 0 \quad (5)$$

$$\frac{\partial H}{\partial \lambda} = x + Gu + c = 0 \quad (2)$$

From eq.(4), $$\lambda^T = -x^T Q \quad (6)$$

From eq.(5), $$u^T R + (-x^T Q G) = 0 \quad (7)$$

From eq.(2), $$x^T = -u^T G^T - c^T \quad (8)$$

Substituting eq.(8) in eq.(7), we get,

$$u^T R + (u^T G^T + c^T) Q G = 0$$

$$u^T (R + G^T Q G) + c^T Q G = 0$$

Hence, $$u = -(R + G^T Q G)^{-1} G^T Q c \quad (9)$$

From eq.(2) and eq.(9),

$$x = G(R + G^T Q G)^{-1} G^T Q c - c$$

Hence, $$x = -(I - G(R + G^T Q G)^{-1} G^T Q) c \quad (10)$$

From eq.(4) and eq (10),

$$\lambda = (Q - Q G (R + G^T Q G)^{-1} G^T Q) c \quad (11)$$

To show the RHS of eq.(11) to be $(Q^{-1} + G R^{-1} G^T)^{-1} c$, we use the following lemma.

MATRIX INVERSION LEMMA.

(This lemma is sometimes referred to as Woodbury's theorem)

Consider $$P = (M^{-1} + H^T R^{-1} H)^{-1} \quad (12)$$

$$= (M^{-1} + H^T R^{-1} H M M^{-1})^{-1}$$

$$= M(I + H^T R^{-1} H M)^{-1}$$

$$= M(I + H^T R^{-1} H M - H^T R^{-1} H M)(I + H^T R^{-1} H M)^{-1}$$

Hence, $$P = M - M H^T R^{-1} H M (I + H^T R^{-1} H M)^{-1} \quad (13)$$

Also, $$HM(I + H^T R^{-1} H M) = (R + H M H^T) R^{-1} H M \quad (14)$$

Premultiplying by $(R + HMH^T)^{-1}$ and postmultiplying by $(I + H^T R^{-1} HM)^{-1}$, eq. (14) yields,

$$(R + HMH^T)^{-1} HM = R^{-1} HM (I + H^T R^{-1} HM)^{-1}$$

Hence, $$P = M - MH^T (R + HMH^T)^{-1} HM \quad (15)$$

Thus, $$(M^{-1} + H^T R^{-1} H)^{-1} = M - MH^T (R + HMH^T)^{-1} HM \quad (16)$$

The matrix inversion lemma [eq. (16)], when used in eq. (11), yields the required result,

$$\lambda = (Q^{-1} + G R^{-1} G^T)^{-1} c \quad (17)$$

To find L_{min}, we could substitute for x and u from eq. (10) and eq. (9) in eq. (1). However,

$$L_{min} = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u, \quad [\text{use eqs. (4), (5)}]$$

$$= -\frac{1}{2} \lambda^T x - \frac{1}{2} \lambda^T G u$$

$$= -\frac{1}{2} \lambda^T (x + Gu), \quad [\text{use eq. (2)}]$$

$$= \frac{1}{2} \lambda^T c, \quad [\text{use eq. (11)}]$$

Hence, $$L_{min} = \frac{1}{2} c^T (Q - QG(R + G^T Q G)^{-1} G^T Q) c \quad (18)$$

Now we will show that $\lambda^T = \frac{\partial J}{\partial c}$ whenever

$$J = L(x,u) \tag{19}$$

and $$f \equiv f_1(x,u) + c = 0 \tag{20}$$

Let us assume that eq. (20) could be solved for x.

Let $x = g(u,c)$ be the solution to eq. (20)

Then $$f \equiv f_1(g(u,c),u) + c = 0 \tag{21}$$

Differentiating eq. (20) or eq (21) with respect to c, we get, $f_x g_c + I = 0$ or $g_c = -f_x^{-1}$

But, $\frac{\partial J}{\partial c} = L_x g_c = -L_x f_x^{-1} \equiv \lambda^T$.

This shows that $\frac{\partial J}{\partial c} = \lambda^T$.

PROBLEM 1.3.5

(a) Performance index is V. The constraint equations are,

$$V^2 = \mu^2 V_r^2 \sin\alpha \sin\theta \tag{1}$$

$$V_r \sin(\alpha+\theta) = W \sin\psi \tag{2}$$

and, $$V_r^2 = V^2 + W^2 + 2VW\cos\psi \tag{3}$$

(Note that $\mu^2 \equiv \frac{C_1}{C_2}$, as in the problem)

State variables are α, V_r and V (4)

Control variable is θ (5)

W, μ and ψ are parameters to be specified in the problem. Note that the choice in (4) and (5) is in conformity with eqs. (1), (2) and (3).

(Once we are given values for W, μ and ψ, specifying the control angle θ determines α, V_r and V through the constraint eqs. (1), (2) and (3))

Now, $$H = V + \lambda_1 (V^2 - \mu^2 V_r^2 \sin\alpha \sin\theta) + \lambda_2 (W \sin\psi - V_r \sin(\alpha+\theta)) + \lambda_3 (V_r^2 - V^2 - W^2 - 2VW\cos\psi) \quad (6)$$

Necessary conditions are eqs. (1), (2), (3) and the following :

$$\frac{\partial H}{\partial \alpha} = -\lambda_1 V^2 \cot\alpha - \lambda_2 V_r \cos(\alpha+\theta) = 0 \quad (7)$$

$$\frac{\partial H}{\partial V_r} = -2\lambda_1 \frac{V^2}{V_r} - \lambda_2 \sin(\alpha+\theta) + 2\lambda_3 V_r = 0 \quad (8)$$

$$\frac{\partial H}{\partial V} = 1 + 2\lambda_1 V - 2\lambda_3 (V + W\cos\psi) = 0 \quad (9)$$

$$\frac{\partial H}{\partial \theta} = -\lambda_1 V^2 \cot\theta - \lambda_2 V_r \cos(\alpha+\theta) = 0 \quad (10)$$

For consistency of eqs. (7) and (10), if $\lambda_1, \lambda_2 \neq 0$, we must have, $$\alpha = \theta \quad (11)$$

(b) With $\alpha = \theta$ and $\psi = 180^\circ$, the constraint eqs. (1), (2) and (3) become,

$$V^2 = \mu^2 V_r^2 \sin^2\theta \quad (12)$$

$$0 = \sin 2\theta \quad (13)$$

and $$V_r^2 = (W - V)^2 \quad (14)$$

respectively.

Eq. (13) is satisfied by $\theta = 90^\circ$ (15)

Substituting (15) and (14) into (12), we get,

$$V = \frac{\mu}{1+\mu} W \tag{16}$$

(c) The performance index in part (a) must now be replaced by $V \cos\psi$, so that

$$H = V\cos\psi + \lambda_1 (V^2 - \mu^2 V_r^2 \sin\alpha \sin\theta) + \lambda_2 (W\sin\psi - V_r \sin(\alpha+\theta)) + \lambda_3 (V_r^2 - V^2 - W^2 - 2VW\cos\psi) \tag{17}$$

The neeessary conditions are, eqs. (1), (2), (3), (5), (6), (8) and the following

$$\frac{\partial H}{\partial \psi} = -V\sin\psi + \lambda_2 W\cos\psi + 2\lambda_3 VW\sin\psi = 0 \tag{18}$$

$$\frac{\partial H}{\partial V} = \cos\psi + 2\lambda_1 V - 2\lambda_3 (V + W\cos\psi) = 0 \tag{19}$$

Eqs. (7) and (10) still imply $\alpha = \theta$. (20)

Eqs (8), (19) and (10) may be solved for λ_1, λ_2 and λ_3. We have,

$$\begin{bmatrix} -2V^2 & -V_r \sin 2\theta & 2V_r^2 \\ -2V & 0 & 2V(1+\frac{W}{V}\cos\psi) \\ V^2\cot\theta & V_r\cos 2\theta & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\psi \\ 0 \end{bmatrix}$$

Substituting the values of λ_2 and λ_3 in eq. (18), we get,

$$\sin\psi = \frac{\left[WV_r\cos\psi\left(\frac{-\cot\theta}{\cos 2\theta}\right) + 2VW\sin\psi\left(1 - \frac{\tan 2\theta}{2\tan\theta}\right)\right]\cos\psi}{2V(V+W)\left(1-\frac{\tan 2\theta}{2\tan\theta}\right) - 2V_r^2} \tag{21}$$

Equations (20) and (21) solved simultaneously with eqs. (1), (2) and (3) determine optimal θ, ψ.

Now assuming $\theta << 1$, eq. (21) simplifies to,

$$\sin\psi \cong \frac{W\cos^2\psi}{2\theta V_r} \tag{21'}$$

Also eqs. (1) and (2) simplify to,

$$V^2 \cong \mu^2 V_r^2 \theta^2 \tag{1'}$$

$$\frac{2\theta}{W} \cong \frac{\sin\psi}{V_r} \tag{2'}$$

Eliminating θ between (21') and (2') yields,

$$\sin^2\psi = \cos^2\psi \text{, that is, } \psi = 45^\circ \tag{22}$$

Substituting (22) and eq. (2') into eq. (1') we get,

$$V^2 = \frac{\mu^2 W^2}{8} \quad \text{or} \quad V = \frac{\mu W}{\sqrt{8}}$$

$$\text{or} \quad V\cos\psi = \frac{\mu W}{4} \tag{23}$$

Substituting eqs. (22) and (23) in eq. (3) we get,

$$V_r^2 = \frac{\mu^2 W^2}{8} + W^2 + \frac{\mu W^2}{2}$$

Or

$$V_r^2 = \left(1 + \frac{\mu}{2} + \frac{\mu^2}{8}\right) W^2 \tag{24}$$

Substituting eqs. (22) and (24) in eq. (2'), we get,

$$\theta \cong \frac{1}{\sqrt{(\mu+2)^2+4}} \tag{25}$$

PROBLEM 1.3.6

Maximize

$$y_f = r(1-\cos\beta_f) \tag{1}$$

Subject to

$$r = \frac{l}{\alpha}\frac{\cos^2\gamma}{\sin\delta} \tag{2}$$

$$\beta_f = \frac{z_c}{l}\frac{\alpha\sin\delta}{\sin\gamma\cos\gamma} \tag{3}$$

$$\gamma = \tan^{-1}\left[\left(\alpha + \frac{\delta^2}{4\alpha}\right)\sec\sigma\right] \quad (4)$$

Form
$$H = r(1-\cos\beta_f) + \lambda_1\left(r - \frac{\ell\cos^2\gamma}{\alpha\sin\sigma}\right) + \lambda_2\left(\beta_f - \frac{z_0}{\ell}\frac{\alpha\sin\sigma}{\sin\gamma\cos\gamma}\right) + \lambda_3\left(\gamma - \tan^{-1}\left[\left(\alpha+\frac{\delta^2}{4\alpha}\right)\sec\sigma\right]\right) \quad (5)$$

Necessary conditions for a maximum are,

$$\frac{\partial H}{\partial r} = 1-\cos\beta_f + \lambda_1 = 0, \quad (6)$$

$$\frac{\partial H}{\partial \beta_f} = r\sin\beta_f + \lambda_2 = 0, \quad (7)$$

$$\frac{\partial H}{\partial \gamma} = \lambda_1\left(\frac{\ell\sin 2\gamma}{\alpha\sin\sigma}\right) + \lambda_2\left(\frac{z_0\alpha\sin\sigma\cos 2\gamma}{\ell(\sin\gamma\cos\gamma)^2}\right) + \lambda_3 = 0, \quad (8)$$

$$\frac{\partial H}{\partial \alpha} = \lambda_1\left(\frac{\ell\cos^2\gamma}{\alpha^2\sin\sigma}\right) - \lambda_2\left(\frac{z_0\sin\sigma}{\ell\sin\gamma\cos\gamma}\right) - \lambda_3\left[\frac{\left(1-\frac{\delta^2}{4\alpha^2}\right)\sec\sigma}{1+\left(\alpha+\frac{\delta^2}{4\alpha}\right)^2\sec^2\sigma}\right] = 0, \quad (9)$$

$$\frac{\partial H}{\partial \sigma} = \lambda_1\left(\frac{\ell\cos^2\gamma\cos\sigma}{\alpha\sin^2\sigma}\right) - \lambda_2\left(\frac{z_0}{\ell}\frac{\alpha\cos\sigma}{\sin\gamma\cos\gamma}\right) - \lambda_3\left[\frac{\left(\alpha+\frac{\delta^2}{4\alpha}\right)\sec\sigma\tan\sigma}{1+\left(\alpha+\frac{\delta^2}{4\alpha}\right)^2\sec^2\sigma}\right] = 0, \quad (10)$$

and equations (2), (3) and (4).

Assume $\alpha, \gamma, \delta \ll 1$, then

$$\text{eq. (9)} \Rightarrow \frac{\ell}{\alpha^2\sin\sigma}\lambda_1 - \frac{z_0\sin\sigma}{\ell\gamma}\lambda_2 - \frac{\left(1-\frac{\delta^2}{4\alpha^2}\right)\sec\sigma}{1}\cdot\lambda_3 = 0$$

$$(i.e)\ 0 = \ell^2\gamma\lambda_1 - z_0\alpha^2\sin^2\sigma\cdot\lambda_2 - \ell\gamma\left(\alpha^2 - \frac{\delta^2}{4}\right)\tan\sigma\cdot\lambda_3 \quad (11)$$

$$\text{eq. (8)} \Rightarrow \frac{\ell\cos\sigma}{\alpha\sin^2\sigma}\lambda_1 - \left[\frac{z_0\alpha\cos\sigma}{\ell\gamma}\right]\lambda_2 - \left[\left(\alpha+\frac{\delta^2}{4\alpha}\right)\sec\sigma\tan\sigma\right]\lambda_3 = 0$$

$$(i.e)\quad 0 = \ell^2\gamma\lambda_1 - z_0\alpha^2\sin^2\sigma.\lambda_2 - \ell\gamma\left(\alpha^2+\frac{\delta^2}{4}\right)\tan^2\sigma.\lambda_3 \quad (12)$$

Comparing eqs. (11) and (12), we get,

$$\alpha^2 - \frac{\delta^2}{4} = \left(\alpha^2+\frac{\delta^2}{4}\right)\tan^2\sigma$$

$$(i.e)\quad (1-\tan^2\sigma)\alpha^2 = \frac{\delta^2}{4}(1+\tan^2\sigma)$$

$$\Rightarrow \qquad \alpha = \frac{\delta}{2}\sqrt{\frac{1+\tan^2\sigma}{1-\tan^2\sigma}} = \frac{\delta}{2\sqrt{\cos 2\sigma}} \quad (13)$$

$$\text{Eq. (4)} \Rightarrow \qquad \gamma = \tan^{-1}\left[\left(\alpha+\frac{\delta^2}{4\alpha}\right)\sec\sigma\right]$$

$$\cong \left(\alpha+\frac{\delta^2}{4\alpha}\right)\sec\sigma$$

$$(i.e)\qquad \gamma = \alpha\left(1+\frac{\delta^2}{4\alpha^2}\right)\sec\sigma \quad (14)$$

Substituting eq. (13) into eq. (14), we get,

$$\gamma = \alpha\left[1+\cos 2\sigma\right]\sec\sigma$$

$$(i.e)\qquad \gamma = 2\alpha\cos\sigma \quad (15)$$

$$\text{Eq. (3)} \Rightarrow \qquad \beta_f = \frac{z_0\alpha\sin\sigma}{\ell\gamma} \quad (16)$$

Eqs. (15) and (16) yield,

$$\frac{\beta_f}{\gamma} = \frac{z_0\alpha}{2\alpha\ell\gamma}\,\frac{\sin\sigma}{\cos\sigma}$$

$$(i.e)\qquad \tan\sigma = \frac{2\beta_f}{(z_0/\ell)} = \frac{2\beta_f}{\zeta} \quad (17)$$

$$\text{Eq. (8)} \Rightarrow \qquad \lambda_1\left(\frac{\ell 2\gamma}{\alpha\sin\sigma}\right) + \lambda_2\left(\frac{z_0\alpha\sin\sigma}{\ell\gamma^2}\right) + \lambda_3 = 0$$

$$(i.e)\quad 2\ell^2\gamma^3\lambda_1 + z_0\alpha^2\sin^2\sigma.\lambda_2 + \ell\alpha\gamma^3\sin\sigma.\lambda_3 = 0 \quad (18)$$

Multiplying eq. (18) by $\sin^2\sigma$ and eq. (11) by $\cos^2\sigma$ and adding (to eliminate λ_3), we get,

$$(2\ell^2\gamma^3\sin^2\sigma + \ell^2\gamma\cos^2\sigma)\lambda_1 + (z_0\alpha^2\sin^4\sigma - z_0\alpha^2\sin^2\sigma\cos^2\sigma)\lambda_2 = 0 \quad (19)$$

Eq. (6) $\Rightarrow$ $$\lambda_1 = -(1-\cos\beta_f) \quad (20)$$

Eq. (7) $\Rightarrow$ $$\lambda_2 = -r\sin\beta_f = -\frac{\ell}{\alpha\sin\sigma}\sin\beta_f \quad (21)$$

Substituting eqs. (20) and (21) into eq. (19), we get,

$$-(2\ell^2\gamma^3\sin^2\sigma + \ell^2\gamma\cos^2\sigma)(1-\cos\beta_f) - (z_0\alpha^2\sin^4\sigma - z_0\alpha^2\sin^2\sigma\cos^2\sigma)\left(\frac{\ell}{\alpha\sin\sigma}\sin\beta_f\right) = 0 \quad (22)$$

Substituting eqs. (13), (14) and (15) into eq. (22) and simplifying, we get, the required result

$$\tan\frac{\beta_f}{2} = \frac{\beta_f}{1 + \frac{4\beta_f^2}{\zeta^2}} \quad \text{where } \zeta \equiv \frac{z_0}{\ell}.$$

PROBLEM 1.3.7

Only an outline of the method of solution (for this problem) is sketched below.

1. It is required to solve the set of five equations given on page 9 of the text (Bryson & Ho), using the numerical values specified

2. Since C_{D_0} and C_{L_α} are functions of V, $\frac{\partial D}{\partial V}$ and $\frac{\partial L}{\partial V}$ will involve the use of $\frac{\partial C_{D_0}}{\partial V}$ and $\frac{\partial C_{L_\alpha}}{\partial V}$.

3. Since the analytical form of C_{D_0} and C_{L_α} are not given, the problem has to be solved numerically

4. Either the first order or second order gradient method may be used.

5. For the gradient method the derivatives may be derived, using the charts and table on pp 16, 17 of the text (Bryson & Ho) and employing finite differences.

PROBLEM 1.3.8

Minimize $$\frac{T}{\beta} = \frac{(C_{D_0} + \eta C_{D_\alpha} \bar{\alpha}^2) mV}{C_{L_\alpha} \bar{\alpha} \sin\sigma} \qquad (1)$$

Subject to $$mg = C_{L_\alpha} \bar{\alpha} \left(\frac{\rho V^2}{2}\right) \delta \cos\sigma \qquad (2)$$

Form $H = \dfrac{(C_{D_0} + \eta C_{D_\alpha} \bar{\alpha}^2)}{C_{L_\alpha} \bar{\alpha} \sin\sigma} mV + \lambda\left(mg - C_{L_\alpha} \bar{\alpha} \left(\frac{\rho V^2}{2}\right) \delta \cos\sigma\right)$

Neeessary conditions for a minimum are,

$$\frac{\partial H}{\partial \bar{\alpha}} = 0 = \frac{C_{D_0} mV}{C_{L_\alpha} \sin\sigma}\left(\frac{-1}{\bar{\alpha}^2}\right) - \lambda C_{L_\alpha}\left(\frac{\rho V^2}{2}\right)\delta\cos\sigma + \frac{\eta mV}{\sin\sigma}, \quad (3)$$

$$\frac{\partial H}{\partial \sigma} = 0 = \frac{(C_{D_0} + \eta C_{L_\alpha} \bar{\alpha}^2)}{C_{L_\alpha} \bar{\alpha}} mV\left(-\frac{\cos\sigma}{\sin^2\sigma}\right) + \lambda C_{L_\alpha} \bar{\alpha}\left(\frac{\rho V^2}{2}\right)\delta\sin\sigma, \quad (4)$$

and eq. (2).

Multiplying eq. (3) by $\bar{\alpha}\sin\sigma$ and eq. (4) by $\cos\sigma$ and adding we get

$$C_{D_0}\left(1 + \frac{\cos^2\sigma}{\sin^2\sigma}\right) = \bar{\alpha}^2 \eta C_{L_\alpha}\left(1 - \frac{\cos^2\sigma}{\sin^2\sigma}\right)$$

$$\Rightarrow \quad \bar{\alpha}^2 = \frac{C_{D_0}}{\eta C_{L_\alpha}(\sin^2\sigma - \cos^2\sigma)} \quad (5)$$

$$\text{or} \quad \alpha^2 = \frac{(\delta/2)^2}{\sin^2\sigma - \cos^2\sigma} \quad (6)$$

Eqs (2) and (5) yield,

$$\cos^2\sigma = \left(\frac{mg}{C_{L_\alpha}\left(\frac{\rho V^2}{2}\right)\delta}\right)^2 \frac{\eta C_{L_\alpha}(\sin^2\sigma - \cos^2\sigma)}{C_{D_0}}$$

$$\Rightarrow \quad \frac{\cos^2\sigma}{\sin^2\sigma - \cos^2\sigma} = \left(\frac{mg}{C_{D_0}\rho V^2}\right)^2 \quad (7)$$

If the numerical values for the RHS of eq. (7) were available, eqs. (6) and (7) yield α and σ.

PROBLEM 1.G (Page 24 of Bryson & Ho)

The solution to this problem may be carried out in the following steps:

1. Guess a set of values for x, u; let these be denoted by x°, u°.
2. Determine λ from $H_x = 0$, call it λ°.
3. Evaluate
$$H_u^\circ \equiv H_u(x^\circ, u^\circ)$$
$$f^\circ \equiv f(x^\circ, u^\circ)$$
4. Linearize the equations $H_u = 0$ and $f = 0$ about x°, u°; this yields
$$H_u^\circ + H_{uu}^\circ \, du + H_{ux}^\circ \, dx = 0 \qquad (A)$$
$$f^\circ + f_u^\circ \, du + f_x^\circ \, dx = 0 \qquad (B)$$
5. Solve eqs. (A) and (B) for dx, du. (It is assumed that x°, u° will not make LHS of eqs. (A) and (B) identically zero.)
6. Repeat steps 1 through 5 for improved values of x, u given by
$$x' = x^\circ + dx \quad \text{and} \quad u' = u^\circ + du$$
until LHS of eqs. (A) and (B) are as close to zero as desired.

PROBLEM 1.7.1

Saddle point is defined by,

$$H(y^0, \lambda) \leq H(y^0, \lambda^0) \leq H(y, \lambda^0) \quad (1)$$

for all $\lambda \geq 0$ and $f(y) \leq 0$.

Note that eq. (1) implies that at (y^0, λ^0) H has a minimum w.r.t y and a maximum w.r.t λ.

The left hand inequality (1) may be written,

$$L(y^0) + \sum_i \lambda_i f_i(y^0) \leq L(y^0) + \sum_i \lambda_i^{(0)} f_i(y^0)$$

$$\Longrightarrow \quad \sum_i \lambda_i f_i(y^0) \leq \sum_i \lambda_i^{(0)} f_i(y^0) \quad (2)$$

Now for any i there are two possibilities:

(i) $f_i(y_0) < 0$

Since the inequality (2) holds for all $\lambda \geq 0$

let us make $\lambda_j = 0$ for all $j \neq i$

then $\lambda_i f_i(y^0) \leq \lambda_i^{(0)} f_i(y^0)$ for all $\lambda_i \geq 0$

Since $f_i(y_0) < 0$, we have

$$\lambda_i \geq \lambda_i^{(0)} \quad \text{for all } \lambda_i \geq 0$$

$$\therefore \quad \lambda_i^{(0)} = 0$$

This is to be expected since the strict inequality $f_i(y^0) < 0$ implies that the constraint is ineffective.

(ii) $f_i(y^0) = 0$ ie, constraint is effective.

In any case, we have

$$\lambda_i^{(0)} f_i(y^0) = 0 \quad \text{for all } i \tag{3}$$

Consider the right hand inequality in (1),

$$L(y^0) + \sum_k \lambda_k^{(0)} f_k(y^0) \leq L(y) + \sum_k \lambda_k^{(0)} f_k(y) . \tag{4}$$

But this is precisely the condition for minimizing $L(y)$ subject to equality constraints. Thus, with (3) and (4) the saddle point theorem follows.

PROBLEM 1.7.3.

The problem is to minimize $L(y_1, y_2)$ subject to a scalar constraint $f(y_1, y_2) \leq 0$.

In the neighborhood of the optimal values y_1^0, y_2^0 a feasible change in y_1, y_2 should not lead to an improvement in the criterion. That is,

$$\delta f \leq 0 \implies \delta L \geq 0 \quad , \text{ and}$$

$$\delta L \leq 0 \implies \delta f \geq 0 .$$

Any improvement in criterion comes only through violation of the constraint. Now, any function increases locally in any direction which makes an angle less than 90° with the gradient direction. If the gradients of f and L do not point in

opposite directions. there will be some direction in which both f and L increase or decrease together thus,

$$\left[\frac{\partial L}{\partial y_1} \cdot \frac{\partial L}{\partial y_2}\right]_{y_1^{(0)}, y_2^{(0)}} = -\lambda \left[\frac{\partial f}{\partial y_1} \cdot \frac{\partial f}{\partial y_2}\right]_{y_1^{(0)}, y_2^{(0)}}$$

ie, $$\frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 \quad , \quad \lambda \geq 0 \quad ,$$

for optimality.

PROBLEM 1.8.1 (Page 31 of text.)

i) Min $L = b^T y$ subject to $A^T y + c \leq 0$.

Let $H = b^T y + \lambda^T (A^T y + c)$, where,

$\lambda = 0$ if $A^T y + c < 0$,

$\lambda > 0$ if $A^T y + c = 0$.

The necessary conditions for a minimum are,

$$A\lambda + b = 0 \quad , \quad (\text{from } \frac{\partial H}{\partial y} = 0) \qquad (1)$$

$$A^T y + c < 0 \quad \text{with } \lambda = 0 \, , \qquad (2)$$

$$A^T y + c = 0 \quad \text{with } \lambda > 0 . \qquad (3)$$

ii) Max $L = c^T \lambda$ subject to $A\lambda + b = 0$ and $\lambda \geq 0$.

Let $H = c^T \lambda + y^T (A\lambda + b) + \mu^T \lambda$, where y and μ are undetermined multipliers. and ,

$$y > 0,$$

$$\mu = 0 \quad \text{if} \quad \lambda > 0,$$

$$\mu > 0 \quad \text{if} \quad \lambda = 0.$$

The necessary conditions for a maximum are,

$$A\lambda + b = 0 \quad \text{, (the constraint itself)}, \tag{4}$$

and $$\frac{\partial H}{\partial \lambda} = 0 \Rightarrow c^T + y^T A + \mu = 0,$$

$$\Rightarrow A^T y + c = -\mu^T.$$

This means,

$$A^T y + c < 0 \quad \text{if} \quad \lambda = 0 \quad (\text{ie. } \mu > 0), \tag{5}$$

$$A^T y + c = 0 \quad \text{if} \quad \lambda > 0 \quad (\text{ie } \mu = 0). \tag{6}$$

A comparison of eqns. (1) – (3) with eqns. (4) – (6) brings out the dual nature of the problems (i) and (ii).

CHAPTER 2

OPTIMIZATION PROBLEMS FOR DYNAMIC SYSTEMS

PROBLEM 2.2.1.

The TPBV can be stated as follows:

$$X(i+1) = \phi(i)X(i) - G(i)\lambda(i+1) \quad (1)$$

$$\lambda(i) = \phi^T(i)\lambda(i+1) + A(i)X(i) \quad (2)$$

where,

$$G(i) \triangleq \Gamma(i)[B(i)]^{-1}\Gamma(i)^T \quad (3)$$

The split boundary conditions are,

$$\lambda(N) = A(N)X(N) \quad ; \quad X(0) \text{ specified.} \quad (4)$$

To apply the "sweep method", let us assume,

$$\lambda(i) = S(i)X(i).$$

Then eq.(1) can be rewritten,

$$\phi(i)X(i) = [I + G(i)S(i+1)]X(i+1), \quad \text{or}$$

$$X(i+1) = [I + \Gamma(i)B^{-1}(i)\Gamma^T(i)S(i+1)]^{-1}\phi(i)X(i) \quad (5)$$

Similarly eq.(2) can be rewritten as,

$$S(i)X(i) = \phi^T(i)S(i+1)[I + G(i)S(i+1)]^{-1}\phi(i)X(i) + A(i)X(i)$$

This yields,

$$S(i) = \phi^T(i)[S^{-1}(i+1) + G(i)]^{-1}\phi(i) + A(i),$$

which can be put in the required form as,

$$S(i) = \phi^T(i)\, M(i+1)\, \phi(i) + A(i) \quad , \text{ where , } \tag{6}$$

$$M(i+1) = \left[S^{-1}(i+1) + \Gamma(i)\, B^{-1}(i)\, \Gamma^T(i)\right]^{-1} \quad , \; i = N-1, N-2, \cdots 0;$$

using the "Matrix Inversion Lemma" this becomes,

$$M(i+1) = S(i+1) - S(i+1)\,\Gamma(i)\left[B(i) + \Gamma^T(i) S(i+1) \Gamma(i)\right]^{-1} \Gamma^T(i)\, S(i+1), \tag{7}$$

Also $\lambda(N) = A(N)X(N) = S(N)X(N)$, so that

$$S(N) = A(N) \tag{8}$$

Now a "backward sweep" of (6) and (7) yields $S(i)$ for $i = N-1, N-2, \cdots, 0$; these can be used in (5) and a "forward sweep" of (5) yields $X(i)$, $i = 1, 2, \cdots N$.

PROBLEM 2.2.2

Performance index: $J = \phi[X(N)] + \sum_{i=0}^{N-1} L^i(x(i), u(i))$

Constraints : $X(i+1) = f^i[x(i), u(i)]$, $i = 0, 1, \cdots, N-1$;

to put this in the framework of section 1.2

let ,

$$\underset{(nN\times 1)}{x} = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} \quad , \quad \underset{(mN\times 1)}{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \quad \text{and} \quad \underset{(nN\times 1)}{f} = \begin{bmatrix} f^0 - x(1) \\ f^1 - x(2) \\ \vdots \\ f^{N-1} - x(N) \end{bmatrix}.$$

Our problem then reduces to,

minimizing $\quad L(x,u) \triangleq J$

subject to $\quad f(x,u) = 0$

By Section 1.2 approach, the necessary conditions are, ($H = L(x,u) + \lambda^T f(x,u)$, $\lambda^T = [\lambda^T(1), \cdots, \lambda^T(N)]$)

(a) $f(x,u) = 0$ (1)

(b) $\frac{\partial H}{\partial x} = 0$, which can be rewritten as,

(i) $\frac{\partial L^i}{\partial x(i)} - \lambda^T(i) \frac{\partial x(i)}{\partial x(i)} + \lambda^T(i+1) \frac{\partial f^i}{\partial x(i)} = 0$,

$i = 1, 2, \cdots, N-1$ (2)

and (ii) $\frac{\partial \phi[x(N)]}{\partial x(N)} - \lambda^T(N) \frac{\partial x(N)}{\partial x(N)}$ (3)

and (c) $\frac{\partial H}{\partial u} = 0$, which can be rewritten as

$\frac{\partial L^i}{\partial u(i)} + \lambda^T(i+1) \frac{\partial f(i)}{\partial u(i)} = 0$,

$i = 0, 1, \cdots N-1$ (4)

Clearly, we have the following correspondences.

eq. (1) $\Leftrightarrow$ eq. (2.2.11) in text.

eq. (2) $\Leftrightarrow$ eq. (2.2.12) in text

eq. (3) $\Leftrightarrow$ eq. (2.2.15) in text

eq. (4) $\Leftrightarrow$ eq. (2.2.13) in text

Also, eq (2.2.14) in text is trivially established.

Thus the necessary conditions of Section 1.2, reduce to that of equations (2.2.11) – (2.2.15).

PROBLEM 2.4.1.

The equations of motion are:

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= a\cos\beta \\ \dot{v} &= a\sin\beta \end{aligned}\right\} \qquad (1)$$

The thrust-direction program is given by:

$$\beta = \begin{cases} \beta_1 , & 0 < t < \frac{T}{2} \\ -\beta_1 , & \frac{T}{2} < t < T \end{cases} \qquad (2)$$

Integrating eq. (1) using eq. (2) we have,

(note that $v(0) = 0 = y(0)$, etc.)

$$v(t) = \int_0^t a \sin\beta(t')\, dt'$$

$$= \begin{cases} at\sin\beta_1 , & 0 < t < \frac{T}{2} \\ -at\sin\beta_1 , & \frac{T}{2} < t < T \end{cases}$$

This clearly implies that $v(T) = 0$.

Hence, $$y(T) = \int_0^T v(t')\, dt' = \frac{aT^2}{4}\sin\beta_1 \equiv h$$

$$\Rightarrow \quad \sin\beta_1 = \frac{4h}{aT^2}$$

Similarly $u(t) = at\cos\beta_1 , \quad 0 < t < T$.

Hence, $u(T) = aT\cos\beta_1$

and, $x(T) = \frac{1}{2} aT^2 \cos\beta_1$.

PROBLEM 2.4.2

Minimize $\underset{\theta(t)}{\int_0^T} y\,(V\cos\theta + u)\,dt$

Subject to, $\dot{x} = V\cos\theta + u \quad , \quad x(T) = x(0)$ (1)

and $\dot{y} = V\sin\theta \quad , \quad y(T) = y(0)$ (2)

Form $H = y(V\cos\theta + u) + \lambda_x (V\cos\theta + u) + \lambda_y V\sin\theta$ (3)

Since H does not involve t explicitly,

H = constant is an integral of the system.

Also, $\dot{\lambda}_x = 0 \quad \Rightarrow \quad \lambda_x = \text{const.}$ (4)

$\dot{\lambda}_y = -(V\cos\theta + u)$ (5)

$0 = -V\sin\theta(\lambda_x + y) + V\cos\theta \cdot \lambda_y$ (6)

Eliminating λ_y between (3) and (6) yields.

$$\cos\theta = \frac{V(y+\lambda_x)}{H - u(\lambda_x + y)} \qquad (7)$$

Since λ_x appears only in conjunction with y its value simply shifts the origin of y co-ordinate. For simplicity choose λ_x so that $\dot{x} = 0$ when $y = 0$. This requires (using (1) and (7)),

$$\lambda_x = \frac{-Hu}{V^2 - u^2} \qquad (8)$$

Substituting (8) into (7) and (7) into (1) and (2) we get,

$$\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{(1-e^2)^{3/2}\, y}{\sqrt{\frac{H^2}{V^2} - y^2(1-e^2)^2}} \quad \text{where } e = \frac{u}{V} \qquad (9)$$

This can be easily integrated (Substitute $z = y^2$ or use Table of integrals, eg. DWIGHT 321.01) to get,

$$x\sqrt{1-e^2} = -\sqrt{\frac{H^2}{V^2} - y^2(1-e^2)^2}$$

$$\Rightarrow \quad x^2 + y^2(1-e^2) = \frac{H^2}{V^2(1-e^2)} \qquad (10)$$

This is clearly the equation to an ellipse with eccentricity e and minor axis parallel to x- axis.

To relate H to the time T, differentiate (7) w.r.t t to get,

$$-\sin\theta \frac{d\theta}{dt} = \frac{\dot{y}}{\frac{H}{V} - \frac{u}{V}(\lambda_x + y)} + \frac{(\lambda_x+y)\frac{u}{V}\dot{y}}{\left[\frac{H}{V} - \frac{u}{V}(\lambda_x+y)\right]^2}$$

$$\Rightarrow \quad -\frac{d\theta}{dt} = \frac{H}{\left[\frac{H}{V} - \frac{u}{V}(\lambda_x+y)\right]^2}$$

$$= \frac{[V + u\cos\theta]^2}{H}$$

Since the path is a closed loop, T is the time for θ to vary from 0 to 2π.

$$\Rightarrow \quad T = \int_0^{2\pi} \frac{H}{V^2} \frac{d\theta}{(1+e\cos\theta)^2}$$

$$\text{or} \qquad T = \frac{H}{V^2} \; \frac{2\pi}{(1-e^2)^{3/2}} \qquad \left[\text{DWIGHT } 859.2\right]$$

$$\Rightarrow \qquad H = \frac{V^2 T \; (1-e^2)^{3/2}}{2\pi}$$

Hence the axes for the ellipse are,

(i) Semi-major axis $\quad a = \frac{VT}{2\pi} \; (1-e^2)^{1/2}$

(ii) Semi-minor axis $\quad b = \frac{VT}{2\pi} \; (1-e^2)$

and so the (maximum) area enclosed by the path is,

$$A_{max} = \pi a b = \frac{V^2 T^2}{4\pi} (1-e^2)^{3/2}.$$

PROBLEM 2.4.3

$$\underset{u}{\text{Minimize}} \int_{-\ell}^{\ell} r\sqrt{1+u^2}\, dx$$

subject to $\frac{dr}{dx} = u \;, \quad r(\ell) = a = r(-\ell)$ (1)

Note that in this problem,

the state variable is r,

the control variable is u, and,

the index variable is x.

Form, $H = r\sqrt{1+u^2} + \lambda u$. (2)

$\Rightarrow H =$ constant.

Also, $H_u = 0 = \frac{ru}{\sqrt{1+u^2}} + \lambda \Rightarrow \lambda = \frac{-ru}{\sqrt{1+u^2}}$ (3)

Substituting (3) in (2),

$$H = r\sqrt{1+u^2} - \frac{ru^2}{\sqrt{1+u^2}}$$

Or, $$r = H\sqrt{1+u^2} \quad (4)$$

$$\Rightarrow u \equiv \frac{dr}{dx} = \sqrt{\frac{r^2-H^2}{H^2}}$$

or, $$\frac{H\,dr}{\sqrt{r^2-H^2}} = dx$$

$$\Rightarrow x + c = H \ln \frac{r+\sqrt{r^2-H^2}}{H}$$

$$= H \cosh^{-1}\left(\frac{r}{H}\right)$$

Hence, $$r = H \cosh\left(\frac{x+c}{H}\right) \quad (5)$$

Using the boundary conditions, $r(\ell) = a = r(-\ell)$, we get $c = 0$, so that,

$$r = H \cosh \frac{x}{H} \quad (6)$$

where $\frac{H}{\ell}$ is determined by $\frac{a}{\ell} \cdot \frac{\ell}{H} = \cosh \frac{\ell}{H}$ (7)

(Refer fig 2.4.7 in the text)

Eq. (7) has two solutions for $0 < \frac{\ell}{a} < .663$

and no solution for $\frac{\ell}{a} > .663$

For $\frac{\ell}{a} > .528$, the minimizing curve is $r \equiv 0$,

i.e., two discs, each with area πa^2.

Hence, for $\frac{l}{a} > .528$, $A_{min} = 2\pi a^2$. and for $\frac{l}{a} < .528$ the minimizing surface of revolution is given by eq. (6).

For $\frac{l}{a} < .528$,

$$A_{min} = 4\pi \int_0^l r\sqrt{1+u^2}\, dx$$

$$= 4\pi H \int_0^l \cosh^2\left(\frac{x}{H}\right) dx$$

$$= 4\pi H^2 \left\{ \frac{\sinh \frac{2l}{H}}{4} + \frac{l}{2H} \right\}$$

$$= \frac{2\pi H^2 \tanh \frac{l}{H}}{\operatorname{sech}^2 \frac{l}{H}} + 2\pi l H$$

$$= 2\pi a^2 \left\{ \frac{\tanh \frac{l}{H}}{\frac{a^2}{H^2}\operatorname{sech}^2 \frac{l}{H}} + \frac{l}{H}\,\frac{H^2}{a^2} \right\}$$

$$= 2\pi a^2 \left(\tanh \frac{l}{H} + \frac{l}{H} \operatorname{sech}^2 \frac{l}{H} \right).$$

(Note: $\frac{a}{H} \operatorname{sech} \frac{l}{H} = 1$.)

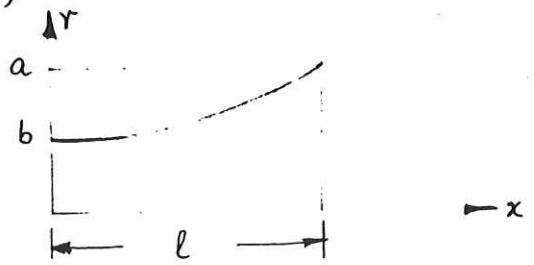
PROBLEM 2.4.4.

We proceed as in previous problem (2.4.3.) and get,

$$r = H \cosh\left(\frac{x+c}{H}\right). \qquad (1)$$

Now, the boundary conditions are different,

and we have, $r(\ell) = a$, $r(0) = b$. (See sketch below.)



$$\Rightarrow \qquad a = H \cosh\left(\frac{\ell+c}{H}\right) , \quad b = H \cosh\frac{c}{H} . \tag{2}$$

Eqn. (2) gives two conditions to determine the constants c and H. Now the situation is almost the same as in problem 2.4.3. All that is needed is to recognize that (2) can be written as,

$$\cosh\frac{c}{H} = \frac{b}{\ell}\cdot\frac{\ell}{H} , \tag{3}$$

$$\cosh\frac{\ell+c}{H} = \frac{a}{\ell}\cdot\frac{\ell}{H} . \tag{4}$$

Now since cosh is an even and monotonic function of its arguments, ($\because \cosh x = 1 + x + \frac{x^2}{2!} + \cdots$), a little argument shows that $0 < |c| < \frac{\ell}{2}$. The rest of the argument is as in previous problem, and for any $\frac{b}{a} < 1$, if $\frac{\ell}{a} > .663$ no solution exists. (If $\frac{\ell+c}{2a} > .528$ then $r = 0$.)

For stationarity, $d\bar{J} = 0 \Rightarrow \dfrac{\partial \bar{\Phi}}{\partial t_f} = -H(t_f)$

$$\equiv -\left(L(t_f) + \frac{\partial \bar{\Phi}}{\partial x}\bigg|_{t_f} \frac{dx}{dt_f}\right)$$

PROBLEM 2.7.2

$$\dot{x} = V(x,y)\cos\theta$$

$$\dot{y} = V(x,y)\sin\theta$$

Form, $H = 1 + \lambda_x V(x,y)\cos\theta + \lambda_y V(x,y)\sin\theta$ (1)

Note, H = constant = 0 (by transversality cond.)

$$\Rightarrow \dot{\lambda}_x = -H_x = -\lambda_x \frac{\partial V}{\partial x}\cos\theta - \lambda_y \frac{\partial V}{\partial x}\sin\theta \quad (*)$$

$$\dot{\lambda}_y = -H_y = -\lambda_x \frac{\partial V}{\partial y}\cos\theta - \lambda_y \frac{\partial V}{\partial y}\sin\theta$$

$$0 = H_\theta = -\lambda_x V\sin\theta + \lambda_y V\cos\theta \quad (2)$$

Using (1) & (2)

$$\lambda_x = -\frac{1}{V(x,y)}\cos\theta \;,\; \lambda_y = -\frac{1}{V(x,y)}\sin\theta \quad (3)$$

Using (*) & (3),

$$\frac{\sin\theta}{V(x,y)}\dot{\theta} + \frac{\cos\theta}{V^2(x,y)}\frac{dV}{dt} = -\lambda_x \frac{\partial V}{\partial x}\cos\theta - \lambda_y \frac{\partial V}{\partial x}\sin\theta \quad (4)$$

But $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial x}\left\{V(x,y)\cos\theta\right\} + \dfrac{\partial V}{\partial y}\left\{V(x,y)\sin\theta\right\}$

Hence eq. (4) can be rewritten as,

$$\frac{\dot{\theta}\sin\theta}{V} + \frac{\cos\theta}{V}\left\{\frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta\right\} = \frac{1}{V}\frac{\partial V}{\partial x}\cos^2\theta + \frac{1}{V}\frac{\partial V}{\partial x}\sin^2\theta$$

$$\Rightarrow \dot{\theta} = \frac{\partial V}{\partial x}\sin\theta - \frac{\partial V}{\partial y}\cos\theta \,.$$

(An alternative method is to demand consistency between $H_\theta = 0$ and $\dot{H}_\theta = 0$, as done in text on p. 78.)

PROBLEM 2.7.3

From previous problem, $\dot{\theta} = \frac{\partial v}{\partial x}\sin\theta - \frac{\partial v}{\partial y}\cos\theta$ (1)

Consider change of variable from t to s where,

$$ds = V\,dt \tag{2}$$

Also, $\dot{x} = V\cos\theta \Rightarrow \frac{dx}{ds}\frac{ds}{dt} = V\cos\theta$

$$\Rightarrow \frac{dx}{ds} = \cos\theta \tag{3}$$

Similarly $\dot{y} = V\sin\theta \Rightarrow \frac{dy}{ds} = \sin\theta$ (4)

In fact (3) & (4) could be written down by geometry.

Consider $\frac{d}{ds}\left(n\frac{dx}{ds}\right) = n\frac{d^2x}{ds^2} + c\frac{dx}{ds}\left(\frac{-\frac{dv}{ds}}{v^2}\right)$ (5)

But $\frac{d^2x}{ds^2} = \frac{d}{ds}(\cos\theta) = -\sin\theta\cdot\frac{d\theta}{ds}$

and (1) $\Rightarrow \frac{d\theta}{ds} = \frac{1}{V}\left\{\frac{\partial v}{\partial x}\sin\theta - \frac{\partial v}{\partial y}\cos\theta\right\}$

Also $\frac{dv}{ds} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$

Hence (5) $\Rightarrow \frac{d}{ds}\left(n\frac{dx}{ds}\right) = n\left\{-\sin\theta\cdot\frac{1}{V}\left(\frac{\partial v}{\partial x}\sin\theta - \frac{\partial v}{\partial y}\cos\theta\right)\right\}$

$$- \frac{c}{v^2}\cos\theta\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)$$

$$= -\frac{\partial v}{\partial x}\cdot\frac{n}{V} = \frac{\partial n}{\partial x}.$$

Similarly, $\frac{d}{ds}\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}$, can be derived.

PROBLEM 2.7.4

In problem 2.7.2, eq. (*) becomes (with $V = V(y)$)

$$\dot{\lambda}_x = 0 \Rightarrow \lambda_x = \text{const.}$$

Now eq. (3) $\Rightarrow$ $\lambda_x = -\frac{1}{V(y)} \cos\theta$

Hence $\frac{\cos\theta}{V(y)} = \text{const.}$, analogous to SNELL'S Law.

PROBLEM 2.7.5.

$$\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{\cos\theta}{\sin\theta} \qquad (1)$$

$$V = V_f\left(1 + \frac{y}{h}\right) \quad ; \; V_f, h \text{ are const.} \qquad (2)$$

Also, $\frac{\cos\theta}{V} = \text{const.}$ (from PROBLEM 2.7.4.)

$\Rightarrow \cos\theta = KV$, where K is a const.

Hence, $$\frac{dx}{dy} = \frac{KV}{\sqrt{1-K^2V^2}} = \frac{K V_f\left(1+\frac{y}{h}\right)}{\sqrt{1-K^2V_f^2\left(1+\frac{y}{h}\right)^2}} \qquad (3)$$

Eq. (3) can be integrated (or use standard tables) to get, $x - x_0 = -\frac{h}{KV_f}\sqrt{1-K^2V_f^2\left(1+\frac{y}{h}\right)^2}$

$$\Rightarrow \left(\frac{x}{h} - \frac{x_0}{h}\right)^2 + \left(\frac{y}{h}+1\right)^2 = \frac{1}{K^2V_f^2} = \text{const.} \qquad (4)$$

(4) shows that the minimum time paths are arcs of circles whose centres lie on the line $y = -h$.

PROBLEM 2.7.6.

$$V = (V_0^2 + 2gy)^{1/2} \equiv V(y)$$

$$\dot{x} = V(y) \cos\theta \qquad (1)$$

$$\dot{y} = V(y) \sin\theta \qquad (2)$$

Form, $H = 1 + \lambda_x V(y) \cos\theta + \lambda_y V(y) \sin\theta \qquad (3)$

Note, $H = \text{const.} = 0$ (by Tranversality $H(t_f) = 0$)

Also, $\dot{\lambda}_x = -H_x = 0 \Rightarrow \lambda_x = \text{const.}, \qquad (4)$

$$\dot{\lambda}_y = -\frac{g}{V(y)} (\lambda_x \cos\theta + \lambda_y \sin\theta) \qquad (5)$$

and $H_\theta = 0 = -\lambda_x V(y) \sin\theta + \lambda_y V(y) \cos\theta \qquad (6)$

Eq.(6) $\Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x} \qquad (7)$

Eq.(3) $\Rightarrow -\frac{1}{V(y)} = \lambda_x \cos\theta + \lambda_y \sin\theta$

$$= \lambda_x \cos\theta + \lambda_x \frac{\sin^2\theta}{\cos\theta}$$

$$\Rightarrow \lambda_x = -\frac{\cos\theta}{V(y)} \qquad (8)$$

Using eqs. (8) and (4) we get,

$$\dot{\lambda}_x = \frac{V \sin\theta\, \dot{\theta} + \cos\theta \frac{dV}{dy}\frac{dy}{dt}}{V^2(y)} = 0$$

$$\Rightarrow \dot{\theta} V \sin\theta + \cos\theta \cdot \frac{g}{V} \cdot V \sin\theta = 0$$

$$\Rightarrow \dot{\theta} = -\frac{g\cos\theta}{V} = g\lambda_x = \text{const.} \qquad (9)$$

To show that the paths are cycloids, we can integrate eqs. (1) & (2) using (9)

$$\frac{dx}{dt} = \frac{dx}{d\theta}\cdot\dot{\theta} \Rightarrow V\cos\theta = \frac{dx}{d\theta}\cdot\dot{\theta}$$

$$\Rightarrow -\frac{g\cos^2\theta}{\dot{\theta}} = \frac{dx}{d\theta}\,\dot{\theta}$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -\frac{g}{\dot{\theta}^2}\cos^2\theta$$

Integrating this equation, (note: $\dot{\theta}$ is const.)

$$x = -\frac{g}{\dot{\theta}^2}\int_{\theta_0}^{\theta}\cos^2\alpha\, d\alpha$$

$$= -\frac{g}{4\dot{\theta}^2}\left[2\alpha + \sin 2\alpha\right]_{\theta_0}^{\theta}$$

But $\dot{\theta} = \text{const.} = -\frac{g\cos\theta}{V} = -\frac{g\cos\theta_0}{V_0}$.

$$\therefore \quad x = -\frac{V_0^2}{4g\cos^2\theta_0}\left[2(\theta-\theta_0) + \sin 2\theta - \sin 2\theta_0\right] \quad (10)$$

Similarly, $\frac{dy}{d\theta} = -\frac{g\sin\theta\cos\theta}{\dot{\theta}^2}$

$$\Rightarrow \quad y = \frac{V_c^2}{4g\cos^2\theta_0}\left[\cos 2\theta - \cos 2\theta_0\right] \quad (11)$$

To show that eqs. (10) and (11) do indeed represent a cycloid, we convert these eqns. to standard form for a cycloid (cusp at origin).

Let $\phi = \pi - 2\theta$, $r = \frac{V_c^2}{4g\cos^2\theta_0}$.

$$\Rightarrow \quad x = -r\left[\pi - 2\theta_0 - \phi + \sin\phi - \sin 2\theta_0\right]$$

$$\Rightarrow \quad x + r\left[\phi_0 - \sin\phi_0\right] = r\left[\phi - \sin\phi\right] \quad (12)$$

Also, $y = r\left[-\cos\phi - 2\cos^2\theta_0 + 1\right]$,

$$\Rightarrow \quad y + 2r\cos^2\theta_0 = r(1-\cos\phi) \qquad (13)$$

Thus defining the quantities r and ϕ as above we find that, the path is a cycloid with cusp at $\left\{-r[\phi_0 - \sin\phi_0], -2r\cos^2\theta_0\right\}$, as defined by eqs. (12) & (13).

In eqs. (10) & (11) θ_0 is to be determined, and for this we use the co-ordinates of the end point B. Let $B \equiv (x_f, y_f)$.

$$\text{Eq. (10)} \Rightarrow -\frac{4g\cos^2\theta_0}{V_c^2}\, x_f = 2(\theta_f - \theta_0) + \sin 2\theta_f - \sin 2\theta_0$$

$$\text{Eq. (11)} \Rightarrow \frac{4g\cos^2\theta_0}{V_c^2}\, y_f = \cos 2\theta_f - \cos 2\theta_0$$

(Since x_f, y_f are non-zero for a meaningful problem)

$$\Rightarrow \frac{\cos 2\theta_0 - \cos 2\theta_f}{y_f} = \frac{2(\theta_f - \theta_0) + \sin 2\theta_f - \sin 2\theta_0}{x_f} \qquad (14)$$

Equation (14) determines θ_f given x_f, y_f, θ_0.

PROBLEM 2.7.7.

Note i) Since gravity is the propelling force on the particle, this means the particle starts from <u>rest</u> (no kinetic energy)

ii) No friction, the system is conservative

$\Rightarrow$ the total energy of the particle is constant.

iii) The gravitational force per unit mass is zero at the centre ($r=0$), is g on the surface ($r = r_0$) and varies linearly for $0 < r < r_0$.

$\therefore$ we take this force as $\frac{g r}{r_0}$

iv) Potential energy at any radius r is given by,

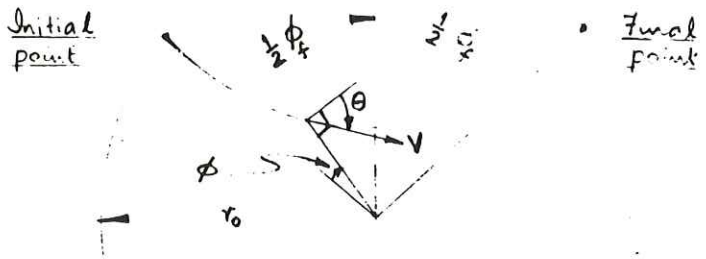
$$\int_0^r \frac{gr}{r_0}\, dr = \frac{1}{2} g \frac{r^2}{r_0}$$

From the above remarks we have,

$$\frac{1}{2} v^2 + \frac{1}{2} g \frac{r^2}{r_0} = \frac{1}{2} g \frac{r_0^2}{r_0}$$

$$\Rightarrow \quad v = \left[g r_0 \left(1 - \frac{r^2}{r_0^2} \right) \right]^{1/2}. \qquad (1)$$

A sketch will be an useful aide, for further work and is drawn below:



The equations of motion are :

$$\dot{r} = -V \sin\theta \quad , \quad r(0) = r_0 \tag{2}$$

$$r\dot{\phi} = V \cos\theta \quad , \quad \phi(0) = 0 \tag{3}$$

As before,

$$H = 1 - \lambda_r V \sin\theta + \lambda_\phi \frac{V}{r} \cos\theta$$

$$\equiv 0 \tag{4}$$

Also, $$\frac{\partial H}{\partial \theta} = -\left[\lambda_r V \cos\theta + \lambda_\phi \frac{V}{r} \sin\theta\right] = 0$$

$$\Rightarrow \lambda_r = -\frac{\lambda_\phi}{r} \tan\theta \quad , \tag{5}$$

and, $$\dot{\lambda}_\phi = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow \lambda_\phi = \text{const.} \tag{6}$$

Eqs. (4) and (5) yield,

$$1 + \lambda_\phi \frac{V}{r}\left(\frac{\sin^2\theta}{\cos\theta} + \cos\theta\right) = 0$$

$$\Rightarrow \cos\theta = -\lambda_\phi \frac{V}{r} \quad , \quad \theta(0) = \frac{\pi}{2} \quad \because V(0) = 0 \tag{7}$$

Using eq. (1) in (7),

$$\cos^2\theta = \lambda_\phi^2 \frac{g}{r_c}\left(\frac{r_0^2}{r^2} - 1\right)$$

$$\Rightarrow \frac{r}{r_c} = \left(1 + \frac{1}{a^2}\cos^2\theta\right)^{-1/2} \quad ; \quad a^2 = \lambda_\phi^2 \frac{g}{r_c} \tag{8}$$

Differentiating eq. (7) w.r.t time,

$$-\sin\theta \cdot \dot{\theta} = -\lambda_\phi \left[\frac{V'}{r} - \frac{V}{r^2}\right]\dot{r} \quad ; \quad V' \triangleq \frac{dV}{dr}$$

$$= +\lambda_\phi \frac{V^2}{r^2} \sin\theta \left[\frac{rV'}{V} - 1\right]$$

But $$V' = \frac{dV}{dr} = -\frac{gr}{Vr_0} \Rightarrow \frac{rV'}{V} = -\frac{gr^2}{V^2 r_0}$$

Hence $$\frac{rV'}{V} - 1 = -\frac{gr_0}{V^2} = -\left(1 - \frac{r^2}{r_c^2}\right)^{-1}$$

$$\therefore \quad \dot{\theta} = \lambda_\phi \frac{\dot{V}^2}{r^2} \frac{1}{1 - r^2/r_0^2} = \lambda_\phi \frac{g}{r_c} \left(\frac{r_c}{r}\right)^2 \qquad (9)$$

Eq. (9) implies that $\lambda_\phi < 0$ ($\because$ we take $a = |\lambda_\phi| \sqrt{\frac{g}{r_c}}$)

Substituting eq. (8) into (9),

$$\dot{\theta} = -a \sqrt{\frac{g}{r_c}} \left(1 + \frac{1}{a^2} \cos^2\theta\right)$$

$$\Rightarrow \sqrt{\frac{g}{r_c}}\, dt = \frac{-a\, d\theta}{a^2 + \cos^2\theta} \qquad (10)$$

Let T = total time taken for the travel,

then by the symmetry and geometry of the problem,

$$\text{eq (10)} \Rightarrow \sqrt{\frac{g}{r_0}} \frac{T}{2} = a \int_0^{\pi/2} \frac{d\theta}{a^2 + \cos^2\theta} \qquad \left[\because \theta(0) = \pi/2,\ \theta\left(\tfrac{T}{2}\right) = 0\right]$$

$$\Rightarrow T \equiv T_{min} = \frac{\pi}{\sqrt{1+a^2}} \sqrt{\frac{r_c}{g}} \qquad (11)$$

Dividing eq (3) by eq. (9),

$$r \frac{d\phi}{d\theta} = \frac{\left[g r_0 \left(1 - \frac{r^2}{r_c^2}\right)\right]^{1/2} \cos\theta}{-a \sqrt{\frac{g}{r_c}} \left(\frac{r_c}{r}\right)^2}$$

$$= -\frac{1}{a} \frac{r^2}{r_0} \sqrt{1 - \frac{r^2}{r_c^2}} \cos\theta \qquad (12)$$

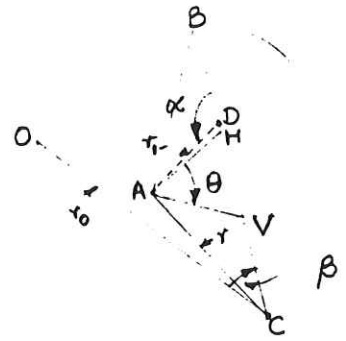
Using eq. (8) in (12)

$$\frac{d\phi}{d\theta} = -\frac{1}{a} \frac{\sqrt{1 - \frac{1}{1 + \frac{1}{a^2}\cos^2\theta}}\, \cos\theta}{\sqrt{1 + \frac{1}{a^2}\cos^2\theta}}$$

$$= -\frac{\cos^2\theta}{a^2 + \cos^2\theta}$$

$$\Rightarrow \phi = \frac{\pi}{2} - \theta - \frac{a}{\sqrt{1+a^2}} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a \tan\theta}{\sqrt{1+a^2}}\right)\right) \qquad (13)$$

We verify below that eqs. (8) & (13) do indeed represent hypocycloids. In what follows, the usual definition of hypocycloid (mentioned in problem statement) is used.



$\angle CAH = \pi/2$

$\angle BAV = \pi/2$

$\angle BAC = \pi - \theta$

$\angle ABD = \frac{\pi}{2} - \frac{\alpha}{2}$

$\angle AVB = \frac{\alpha}{2}$

Let $\angle BDA = \alpha$. Then $\angle ABD = \frac{\pi}{2} - \frac{\alpha}{2}$, $\angle AVB = \frac{\alpha}{2}$

Also from geometry, $\angle CAH = \frac{\pi}{2} = \angle BAV$, $\angle BAC = \pi - \theta$.

We can write down 4 eqns. for α, β, r and ϕ in terms of r_0, r_1 and θ as follows:

$$\alpha r_1 = \beta r_0 \quad \text{(No slip rolling condition)} \tag{1}$$

$$\theta - \beta + \phi = \frac{\pi}{2} - \frac{\alpha}{2} \quad \text{(Sum of } \angle s \text{ of } \Delta ABC = \pi) \tag{2}$$

$$\frac{\sin\theta}{r_0} = \frac{\cos\alpha/2}{r} \quad \text{(Sine rule for } \Delta ABC) \tag{3}$$

$$\frac{\sin\alpha}{r} = \frac{\sin(\beta - \phi)}{r_1} \quad \text{(Sine rule for } \Delta ADC) \tag{4}$$

Substituting (2) into (4),

$$\frac{r_1}{r}\sin\alpha = \sin\left(\theta + \frac{\alpha}{2} - \frac{\pi}{2}\right) = -\cos\left(\theta + \frac{\alpha}{2}\right)$$

$$\Rightarrow \quad 2\frac{r_1}{r}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\theta\sin\frac{\alpha}{2} - \cos\theta\cos\frac{\alpha}{2} \qquad (5')$$

$$= \frac{r_0}{r}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} - \cos\theta\cos\frac{\alpha}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2} = \frac{\operatorname{ctn}\theta}{1 - 2\frac{r_1}{r_0}} \qquad (6')$$

Substituting (6) into (3), (with $b \triangleq 1 - \frac{2r_1}{r_0}$)

$$\frac{r}{r_0} = \frac{\cos\frac{\alpha}{2}}{\sin\theta} = \frac{b}{\sin\theta\sqrt{b^2+\operatorname{ctn}^2\theta}} = \frac{b}{\sqrt{b^2+(1-b^2)\cos^2\theta}}$$

$$\Rightarrow \frac{r}{r_0} = \frac{1}{\sqrt{1+\frac{1}{a^2}\cos^2\theta}} \quad \text{where} \quad a^2 = \frac{b^2}{1-b^2}. \qquad (7')$$

Substituting (1) into (2) to eliminate β,

$$\phi = \frac{\pi}{2} - \theta + \left(\frac{2r_1}{r_0} - 1\right)\frac{\alpha}{2} \qquad (8')$$

Using (6) in (8),

$$\phi = \frac{\pi}{2} - \theta + b\left[\tan^{-1}\left(\frac{b}{\operatorname{ctn}\theta}\right) - \frac{\pi}{2}\right]$$

$$\Rightarrow \phi = \frac{\pi}{2} - \theta - \frac{a}{\sqrt{1+a^2}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a\tan\theta}{\sqrt{1+a^2}}\right)\right) \qquad (9')$$

Eqs. (7') & (9') are identical to eqs. (8) & (13). Hence, the minimum time paths are hypocycloids. Note that $\frac{r_1}{r_0} = \frac{\beta}{\alpha} = \text{const.} = \frac{\phi_f}{2\pi}$.

PROBLEMS 2.7.8 & 2.7.9 (Figs 2.4.1, 2.4.2 in text.)

$$\dot{u} = a\cos\beta \tag{1}$$

$$\dot{v} = a\sin\beta \tag{2}$$

$$\dot{x} = u \tag{3}$$

$$\dot{y} = v \tag{4}$$

Form, $H = 1 + \lambda_u a\cos\beta + \lambda_v a\sin\beta + \lambda_x u + \lambda_y v$ (5)

Necessary conditions :

(i) $\dot{\lambda}_u = -\lambda_x$, $\dot{\lambda}_v = -\lambda_y$, $\dot{\lambda}_x = 0 = \lambda_y$ (6)

$$\Rightarrow \lambda_x = \text{const.} = C_1 \;,\; \lambda_y = C_2$$
$$\lambda_u = -C_1 t + C_3 \;,\; \lambda_v = -C_2 t + C_4 \Big\} \tag{7}$$

(ii) $H_\beta = 0 \Rightarrow \tan\beta = \dfrac{\lambda_v}{\lambda_u} = \dfrac{-C_2 t + C_4}{-C_1 t + C_3}$ (8)

(iii)
$$\left.\begin{aligned} u(0) &= 0 \;,\; u(T) = V \\ v(0) &= 0 \;,\; v(T) = 0 \\ x(0) &= 0 \;,\; \lambda_x(T) = 0 \text{ since } x(T) \text{ is unspecified} \\ y(0) &= 0 \;,\; \lambda_y(T) \text{ not specified}, \; y(T) = h \end{aligned}\right\} \tag{9}$$

Transversality condition :

$$H(t=T) = 0 \Rightarrow (\lambda_u a\cos\beta + \lambda_v a\sin\beta)_{t=T} = -1 \tag{10}$$

Also, $H(t) = \text{const} = H(T) = 0$, since H is not an explicit function of time.

Now, $\lambda_x = C_1 = \lambda_x(T) = 0$, so that

eq. (8) $\Rightarrow$ $\tan\beta = \tan\beta_0 - ct$ where, (11)

$$\tan\beta_0 = \frac{C_4}{C_3} \;,\; c = \frac{C_2}{C_3}$$

Eq. (11) $\Rightarrow$ $\sec^2\beta \cdot \dot{\beta} = -c$

Or, $\dot{\beta} = -c\cos^2\beta$ (12)

Hence, eqs. (1) — (4) can be rewritten as,

$$\frac{du}{d\beta} = -\frac{a}{c}\sec\beta \quad , \quad u(\beta_0) = 0 \tag{13}$$

$$\frac{dv}{d\beta} = -\frac{a}{c}\tan\beta\sec\beta \quad , \quad v(\beta_0) = 0 \tag{14}$$

$$\frac{dx}{d\beta} = -\frac{u}{c}\sec^2\beta \quad , \quad x(\beta_0) = 0 \tag{15}$$

$$\frac{dy}{d\beta} = -\frac{v}{c}\sec^2\beta \quad , \quad y(\beta_0) = 0 \tag{16}$$

Moreover $t \equiv t(\beta) = \dfrac{\tan\beta_0 - \tan\beta}{c}$. (17)

Solving (13) — (16),

$$u = \frac{a}{c}\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta} \tag{18}$$

$$v = \frac{a}{c}(\sec\beta_0 - \sec\beta) \tag{19}$$

$$x = -\frac{a}{c^2}\int_{\beta_0}^{\beta}\sec^2\alpha\log\left(\frac{\tan\beta_0 + \sec\beta_0}{\tan\alpha + \sec\alpha}\right)d\alpha$$

Integrating by parts,

$$\frac{c^2x}{a} = (\tan\beta - \tan\beta_0)\log\left(\frac{\tan\beta + \sec\beta}{\tan\beta_0 + \sec\beta_0}\right) - \int_{\beta_0}^{\beta}(\tan\alpha - \tan\beta_0)\sec\alpha\, d\alpha$$

$$\Rightarrow x = \frac{a}{c^2}\left[\sec\beta_0 - \sec\beta - \tan\beta\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\right] \tag{20}$$

Similarly, $$y = -\frac{a}{c^2}\int_{\beta_0}^{\beta} (\sec\beta_0 - \sec\alpha)\sec^2\alpha\, d\alpha$$
$$= -\frac{a}{c^2}\sec\beta_0(\tan\beta - \tan\beta_0) + \frac{a}{c^2}\int_{\beta_0}^{\beta}\sec^3\alpha\, d\alpha$$

Let $$I = \int_{\beta_0}^{\beta}\sec^3\alpha\, d\alpha = \int_{\beta_0}^{\beta}\sec\alpha\cdot\sec^2\alpha\, d\alpha$$

$$= \sec\alpha\cdot(\tan\alpha - \tan\beta_0)\Big|_{\beta_0}^{\beta} - \int_{\beta_0}^{\beta}(\tan\alpha - \tan\beta_0)\sec\alpha\tan\alpha\, d\alpha$$

$$= \sec\beta(\tan\beta - \tan\beta_0) - \Big[\int_{\beta_0}^{\beta}\sec^3\alpha\, d\alpha - \int_{\beta_0}^{\beta}\sec\alpha\, d\alpha - \tan\beta_0\int_{\beta_0}^{\beta}\sec\alpha\tan\alpha\, d\alpha\Big]$$

$$\Rightarrow 2I = \sec\beta(\tan\beta - \tan\beta_0) + \Big[\log(\tan\alpha + \sec\alpha) + \tan\beta_0\sec\alpha\Big]_{\beta_0}^{\beta}$$
$$= \sec\beta(\tan\beta - \tan\beta_0) + \log\frac{\tan\beta + \sec\beta}{\tan\beta_0 + \sec\beta_0} + \tan\beta_0(\sec\beta - \sec\beta_0)$$

Hence $$y = \frac{a}{2c^2}\Big[\sec\beta_0(\tan\beta_0 - \tan\beta) - \tan\beta(\sec\beta_0 - \sec\beta) - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta + \sec\beta}\Big] \qquad (21)$$

Now $V(T) = 0 \Rightarrow \frac{a}{c}(\sec\beta_0 - \sec\beta(T)) = 0$

so that, $c = \frac{\tan\beta_0 - \tan\beta(T)}{T} = \frac{2\tan\beta_0}{T}$ (22)

(Note that we chose $\beta_0 = -\beta(T)$ for $c \neq 0$)

Also, $H = 0 = H_\beta = a(\lambda_v\cos\beta - \lambda_u\sin\beta)$

$$= 1 + \lambda_u a\cos\beta + \lambda_v a\sin\beta + \lambda_x u + \lambda_y v$$

at $t = 0$, $\Rightarrow a\lambda_v(0)\cos\beta_0 - a\lambda_u(0)\sin\beta_0 = 1 + \lambda_u(0)a\cos\beta_0 + \lambda_v(0)a\sin\beta_0$

$$\Rightarrow a\lambda_u \sin\beta_0 - a\lambda_u \sin\beta_0 = 1 + \lambda_u a\cos\beta_0 + \lambda_u\, a\,\frac{\sin^2\beta_0}{\cos\beta_0}$$

$$\Rightarrow \qquad \lambda_u \;=\; -\frac{\cos\beta_0}{a} \;\equiv\; C_3 \qquad (23)$$

(Note : $\lambda_u(0) = \lambda_u = C_3$, $\because\ \lambda_x = C_1 = 0$)

Also, $1 + \lambda_v(0)\,\frac{a\cos^2\beta_0}{\sin\beta_0} + \lambda_v(0)\, a\sin\beta_0 = 0$

$$\Longrightarrow \qquad \lambda_v(0) \;=\; -\frac{\sin\beta_0}{a} \;\equiv\; C_4 \qquad (24)$$

Now $\qquad c \;=\; \frac{c_2}{c_3} \;=\; \frac{2\tan\beta_0}{T}$

$$\Longrightarrow \qquad c_2 \;=\; -\frac{2\sin\beta_0}{aT}$$

Hence ,

$$\left.\begin{aligned} \lambda_u &\equiv C_3 = -\frac{\cos\beta_0}{a} \\ \lambda_v &\equiv -c_2 t + c_4 = -\frac{\sin\beta_0}{a}\left(1 - \frac{2t}{T}\right) \\ \lambda_x &\equiv C_1 = 0 \\ \lambda_y &\equiv C_2 = -\frac{2\sin\beta_0}{aT} \end{aligned}\right\} \qquad (25)$$

The constants β_0 and T (minimum time) are determined from ,

$$y(T) \equiv h \;=\; \frac{a}{2c^2}\left[2\tan\beta_0 \sec\beta_0 - \log\frac{\tan\beta_0 + \sec\beta_0}{\sec\beta_0 - \tan\beta_0}\right] \qquad (26)$$

$$U \;\equiv u(T) = \frac{a}{c}\left[\log\frac{\tan\beta_0 + \sec\beta_0}{\sec\beta_0 - \tan\beta_0}\right] \qquad (27)$$

In order to put (26) & (27) in the required form, we proceed as follows :

(i) $\tan\beta_0 + \sec\beta_0 = \dfrac{2\tan\frac{\beta_0}{2}}{1-\tan^2\frac{\beta_0}{2}} + \dfrac{1}{\cos^2\frac{\beta_0}{2} - \sin^2\frac{\beta_0}{2}}$

$= \dfrac{2\tan\frac{\beta_0}{2} + \sec^2\frac{\beta_0}{2}}{1-\tan^2\frac{\beta_0}{2}}$

$= \dfrac{\left(1+\tan\frac{\beta_0}{2}\right)^2}{1-\tan^2\frac{\beta_0}{2}}$

$= \dfrac{1+\tan\frac{\beta_0}{2}}{1-\tan\frac{\beta_0}{2}}$

$\Rightarrow \tan\beta_0 + \sec\beta_0 = \tan\left(\frac{\pi}{4} + \frac{\beta_0}{2}\right)$ (28)

(ii) $\tan\left(\frac{\pi}{4} - \frac{\beta_0}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\beta_0}{2}\right)$

$= \cot\left(\frac{\pi}{4} + \frac{\beta_0}{2}\right)$ (29)

(iii) In (28) replace β_0 by $-\beta_0$,

$\therefore -\tan\beta_0 + \sec\beta_0 = \tan\left(\frac{\pi}{4} - \frac{\beta_0}{2}\right)$ (30)

Hence, $\dfrac{aT}{U} = \dfrac{\tan\beta_0}{\frac{1}{2}\log\frac{\tan\beta_0+\sec\beta_0}{\sec\beta_0-\tan\beta_0}} = \dfrac{\tan\beta_0}{\log\tan\left(\frac{\pi}{4}+\frac{\beta_0}{2}\right)}$ (31)

and $\dfrac{4ah}{U^2} = \dfrac{2\left[2\tan\beta_0\sec\beta_0 - \log\left(\frac{\tan\beta_0+\sec\beta_0}{\sec\beta_0-\tan\beta_0}\right)\right]}{\left[\log\left(\frac{\tan\beta_0+\sec\beta_0}{\sec\beta_0-\tan\beta_0}\right)\right]^2}$

$= \dfrac{\tan\beta_0\sec\beta_0 - \frac{1}{2}\log\left(\frac{\tan\beta_0+\sec\beta_0}{\sec\beta_0-\tan\beta_0}\right)}{\left[\frac{1}{2}\log\left(\frac{\tan\beta_0+\sec\beta_0}{\sec\beta_0-\tan\beta_0}\right)\right]^2}$

$\Rightarrow \dfrac{4ah}{U^2} = \dfrac{\tan\beta_0\sec\beta_0 - \log\tan\left(\frac{\pi}{4}+\frac{\beta_0}{2}\right)}{\left[\log\tan\left(\frac{\pi}{4}+\frac{\beta_0}{2}\right)\right]^2}$. (32)

PROBLEM 2.7.10.

The relevant equations are,

$$\dot{u} = a\cos\beta \quad , \quad u(0) = u_0 \quad , \quad \lambda_u(T) = 0 \tag{1}$$

$$\dot{v} = a\sin\beta \quad , \quad v(0) = v_0 \quad , \quad \lambda_v(T) = 0 \tag{2}$$

$$\dot{x} = u \quad , \quad x(0) = x_0 \quad , \quad x(T) = 0 \tag{3}$$

$$\dot{y} = v \quad , \quad y(0) = y_0 \quad , \quad y(T) = 0 \tag{4}$$

Also, $H = 1 + \lambda_u a\cos\beta + \lambda_v a\sin\beta + \lambda_x u + \lambda_y v$

$= \text{const.} = H(t=T) = 0$, by transversality.

$$H_\beta = 0 \implies \tan\beta = \frac{\lambda_v}{\lambda_u} \tag{5}$$

The co-state equations are,

$$\dot{\lambda}_x = \dot{\lambda}_y = 0 \quad , \quad \dot{\lambda}_u = -\lambda_x \quad , \quad \dot{\lambda}_v = -\lambda_y \tag{6}$$

Hence, $$\tan\beta = \frac{-\lambda_y (t-T)}{-\lambda_x (t-T)} = \frac{\lambda_y}{\lambda_x} = \text{const.} \tag{7}$$

Now, it is easy to solve the equations of motion, and we get,

$$x = x_0 + u_0 t + \frac{at^2}{2}\cos\beta \tag{8}$$

$$y = y_0 + v_0 t + \frac{at^2}{2}\sin\beta \tag{9}$$

Using $x(T) = 0 = y(T)$, we get, two eqns. for β & T.

$$0 = x_0 + u_0 T + \frac{aT^2}{2}\cos\beta \tag{10}$$

$$0 = y_0 + v_0 T + \frac{aT^2}{2}\sin\beta \tag{11}$$

PROBLEM 2.7.11.

This problem is straightforward. See Appendix C.

PROBLEM 2.7.12.

The solution is obvious.

PROBLEM 2.7.13.

Integration of $\ddot{u}$ and $\ddot{x}$ equations yields results identical to problem 2.7.9. Integration of $\ddot{y}$ needs the extra term, $\int -g\,dt = -gt$. and the constant of integration is same as before. Integration of $\dot{y}$ needs the extra term. $\int -gt\,dt = -gt^2/2$. Thus,

$$v(T) = 0 = \frac{a}{c}(\sec\beta_0 - \sec\beta_f) - gT \tag{1}$$

$$y(T) = h = \frac{a}{2c^2}\left[(\tan\beta_0 - \tan\beta_f)\sec\beta_0 - (\sec\beta_0 - \sec\beta_f)\tan\beta_f - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f}\right] - \frac{1}{2}gT^2 \tag{2}$$

We still have,

$$c = \frac{\tan\beta_0 - \tan\beta_f}{T} \tag{3}$$

$$U = \frac{a}{c}\log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f} \tag{4}$$

Eliminating cT from (1) & (3),

$$\frac{a}{g} = \frac{\tan\beta_0 - \tan\beta_f}{\sec\beta_0 - \sec\beta_f}. \tag{5}$$

Eq. (2) $\Rightarrow$

$$c^2 = \frac{a}{2h}\left[\tan\beta_0\sec\beta_f - \tan\beta_f\sec\beta_0 - \log\frac{\tan\beta_0 + \sec\beta_0}{\tan\beta_f + \sec\beta_f} + (\sec\beta_0 - \sec\beta_f)(\tan\beta_0 - \tan\beta_f)\right] - \frac{1}{2}g\frac{c^2T^2}{h} \tag{6}$$

Using (1) and (3) the last two terms of (6) cancel and

$$\frac{2ah}{U^2} = \frac{\tan\beta_0 \sec\beta_f - \tan\beta_f \sec\beta_0 - \log\dfrac{\tan\beta_0+\sec\beta_0}{\tan\beta_f+\sec\beta_f}}{\left[\log\dfrac{\tan\beta_0+\sec\beta_0}{\tan\beta_f+\sec\beta_f}\right]^2}$$

$$\frac{aT}{U} = \frac{\tan\beta_0 - \tan\beta_f}{\log\left(\dfrac{\tan\beta_0+\sec\beta_0}{\tan\beta_f+\sec\beta_f}\right)}$$

PROBLEM 2.7.14.

a) $$\frac{dx}{dt} = u \quad , \quad x_0^2 + t_0^2 = 1 \qquad (1)$$

$$\min. \quad J = \int_{t_0}^{2} \sqrt{1+u^2}\, dt \qquad (2)$$

The necessary conditions are,

$$H_u = 0 \Rightarrow \frac{u}{\sqrt{1+u^2}} + \lambda = 0 , \qquad (3)$$

$$\dot{\lambda} = 0 \Rightarrow \lambda = \text{const} = 0 , \because x(T) \text{ is}$$

free, so that $\lambda(T) = 0$.

Also, $\lambda(0) = 2\mu x_0 = 0$. (4)

The transversality condition (for t_0), is,

$$2\mu t_0 - \sqrt{1+u_0^2} - \lambda_0 u_0 = 0 \qquad (5)$$

(Note that $\bar{J} = \mu(x_0^2 + t_0^2 - 1) + J$.) :

Since $\lambda(t) = 0$, eq. (3) $\Rightarrow u = 0$. Hence (5),

$\Rightarrow 2\mu = \frac{1}{t_0}$. (In fact we can only say $2\mu t_0 \neq 0$, but this with $2\mu x_0 = 0$ and $x_0^2 + t_0^2 = 1$ yields $t_0 \neq 0$ and $x_0 = 0$)

Thus the extremal is a straight line joining $x_0 = 0, t_0 = 1$ to $x = 0, t = 2$. (The obvious solution)

b) Here both x_0 and x_f are free.

$$\dot{x} = u \tag{1}$$

$$\bar{J} = \mu_f(-x_f + t_f - 2\sqrt{2}) + \mu_0(x_0^2 + t_0^2 - 1) + \int_{t_0}^{t_f} (\sqrt{1+u^2} + \lambda u)\, dt \tag{2}$$

Remembering that there will be <u>two</u> transversality conditions, the necessary conditions are easily written down as below: (Note that $H = \sqrt{1+u^2} + \lambda u$ and $\bar{J}$ is augmented criterion.)

$$H_u = 0 \Rightarrow \frac{u}{\sqrt{1+u^2}} + \lambda = 0 \tag{3}$$

$$\dot{\lambda} = 0, \quad \lambda(t_f) = -\mu_f = \lambda(t_0) = -2\mu_0 x_0 \tag{4}$$

The two transversality conditions are:

$$H(t_0) = 2\mu_0 t_0 = \sqrt{1+u_0^2} + \lambda u_0 \tag{5}$$

$$H(t_f) = -\mu_f = \sqrt{1+u_f^2} + \lambda u_f \tag{6}$$

Clearly (4) $\Rightarrow$ $\lambda(t) = \text{const} = -2\mu_0 x_0 = -\mu_f$ (7)

(3) $\Rightarrow$ $u(t) = \text{const} \Rightarrow \frac{u}{\sqrt{1+u^2}} = -\lambda = 2\mu_0 x_0$ (8)

(6) $\Rightarrow$ $-2\mu_0 x_0 = \sqrt{1+u^2} + u(-2\mu_0 x_0)$

$\Rightarrow$ $-2\mu_0 x_0 (1-u) = \sqrt{1+u^2}$ (9)

(8) & (9) $\Rightarrow$ $\frac{-u}{\sqrt{1+u^2}}(1-u) = \sqrt{1+u^2} \Rightarrow u = -1$ (10)

$\therefore$ (3) $\Rightarrow$ $\lambda = -2\mu_0 x_0 = \frac{1}{\sqrt{2}}$ (11)

(5) $\Rightarrow$ $2\mu_0 t_0 = \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ (12)

Using (11), (12) and $x_0^2 + t_0^2 = 1$, we find,

$$x_0 = -t_0 = -1/\sqrt{2}.$$

Integrating (1) and using $-x_f + t_f = 2\sqrt{2}$, we get $x_f = -t_f = -\sqrt{2}$.

CHAPTER 4

OPTIMAL FEEDBACK CONTROL.

PROBLEM 4.1.1.

It was shown in 2.7.4 that, $\frac{\cos\theta}{V} = \text{const.} = \frac{\cos\theta_f}{V_f}$. (1)

But, $V = V_f\left(1+\frac{y}{h}\right) \Rightarrow \quad \frac{y}{h} = \frac{\cos\theta}{\cos\theta_f} - 1$, using (1). (2) (3)

From problem, 2.7.2,

$$\dot{\theta} = -\frac{\partial V}{\partial y}\cos\theta$$

$$= -\frac{V_f}{h}\cos\theta \qquad (4)$$

Integrating eq. (4),

$$\frac{V_f}{h}(t_f - t) = \log\frac{\sec\theta + \tan\theta}{\sec\theta_f + \tan\theta_f}. \qquad (5)$$

Now, $\dot{x} = V\cos\theta$

$$\Rightarrow -\frac{dx}{d\theta}\frac{V_f}{h}\cos\theta = \frac{V_f}{\cos\theta_f}\cos^2\theta$$

or, $-dx = \frac{h}{\cos\theta_f}\cos\theta\, d\theta$

On integration this yields, $\frac{x}{h} = \tan\theta_f - \sin\theta\sec\theta_f$. (6)

Eqns. (3), (6) and (5) are the parametric eqns. for the minimum time paths and the time-to-go. Eliminating θ between (3) and (6),

$$\left(\frac{x}{h} - \tan\theta_f\right)^2 + \left(\frac{y}{h}+1\right)^2 = \sec^2\theta_f. \qquad (7)$$

Here we have another derivation for problem 2.7.5, and eq. (7) represents a one-parameter

family of circles with centres on $y = -h$, and passing through the origin $x = 0 = y$. (See accompanying figure)

To get contours of constant Θ, let us eliminate θ_f between (3) & (6); this yields

$$\sec\theta_f = \left(1 + \frac{y}{h}\right)\sec\theta ,$$

$$\Rightarrow \left[\frac{x}{h} + \left(1 + \frac{y}{h}\right)\tan\theta\right]^2 = \left[\left(1 + \frac{y}{h}\right)\sec\theta\right]^2 - 1$$

$$\text{Or } \left(\frac{x}{h}\right)^2 + 2\tan\theta\,\frac{x}{h}\,\frac{y}{h} - \left(\frac{y}{h}\right)^2 + 2\tan\theta\cdot\frac{x}{h} - 2\frac{y}{h} = 0 \quad (8)$$

Equation (8) represents a family of hyperbolas (parameter θ), with axes oblique to the co-ordinate axes.

[Note: A check of equation (8) may be made using, for e.g., CRC Standard mathematical Tables, p 366 :

comparing (8) with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

we find $B^2 - 4AC = +4\sec^2\theta > 0$

$\Rightarrow$ eq. (8) represents hyperbolas with axes oblique to the co-ordinate axes Alternatively, a comparison with the general quadratic equation in x & y, viz., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ could be made and Δ, J could be

determined as follows:

$$\Delta \triangleq \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = -\sec^2\theta = 0$$

$$J \triangleq \begin{vmatrix} a & h \\ h & b \end{vmatrix} = -\sec^2\theta < 0$$

This, from the CRC table on p. 367, shows that eq. (8) represents a hyperbola.]

Moreover, the hyperbolas, eq. (8) pass through the origin, and the centres of the conic (hyperbolas here) are given by,

$$ax' + hy' + g = 0 \quad , \quad hx' + by' + f = 0$$

which for our case reduce to,

$$\left.\begin{aligned} x' + y'\tan\theta + \tan\theta &= 0 \\ x'\tan\theta + (-y') + (-1) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x' &= 0 \\ y' &= -1 \end{aligned}$$

Remembering that we are dealing with normalized variables we have

$$\left.\begin{aligned} x' \equiv \frac{x_c}{h} = 0 &\Rightarrow x_c = 0 \\ y' \equiv \frac{y_c}{h} = -1 &\Rightarrow y_c = -h \end{aligned}\right\} \quad (9)$$

(The subscript c denotes centre.). The equation to the axes of the conic are,

$$\left.\begin{aligned} (y - y_c) &= m(x - x_c) \quad , \quad (y - y_c) = -\frac{1}{m}(x - x_c) \\ \text{where} \quad m &= \frac{\sec\theta - 1}{\tan\theta} \end{aligned}\right\} \quad (10)$$

To find the isochrones, it is convenient to change the independent variable to $\tau \equiv t_f - t$ (time-to-go) so that the "initial" ($\tau = 0$) condition is $x = y = 0$.

We will rewrite eq.(7) (for ease of manipulation) as,

$$(x+c)^2 + (y+h)^2 = c^2 + h^2 \qquad (11)$$

Hence,

$$\frac{dx}{dt} = V_f\left(1 + \frac{y}{h}\right)\cos\theta$$

$$\Rightarrow \quad \frac{dx}{d\tau} = \frac{V_f\left[c^2+h^2-(x+c)^2\right]}{h\sqrt{c^2+h^2}}, \quad x > 0 \qquad (12)$$

In arriving at eq.(12) we made use of (1) & (11).

Integrating (12) with $x(0) = 0$, we get.

$$\frac{\sqrt{c^2+h^2} + x + c}{\sqrt{c^2+h^2} - (x+c)} = \frac{\sqrt{c^2+h^2} + c}{\sqrt{c^2+h^2} - c}\; e^{\frac{2V_f}{h}\tau} \qquad (13)$$

Solving (11) for c and using this result in (13) we get, after a lot of algebraic manipulation,

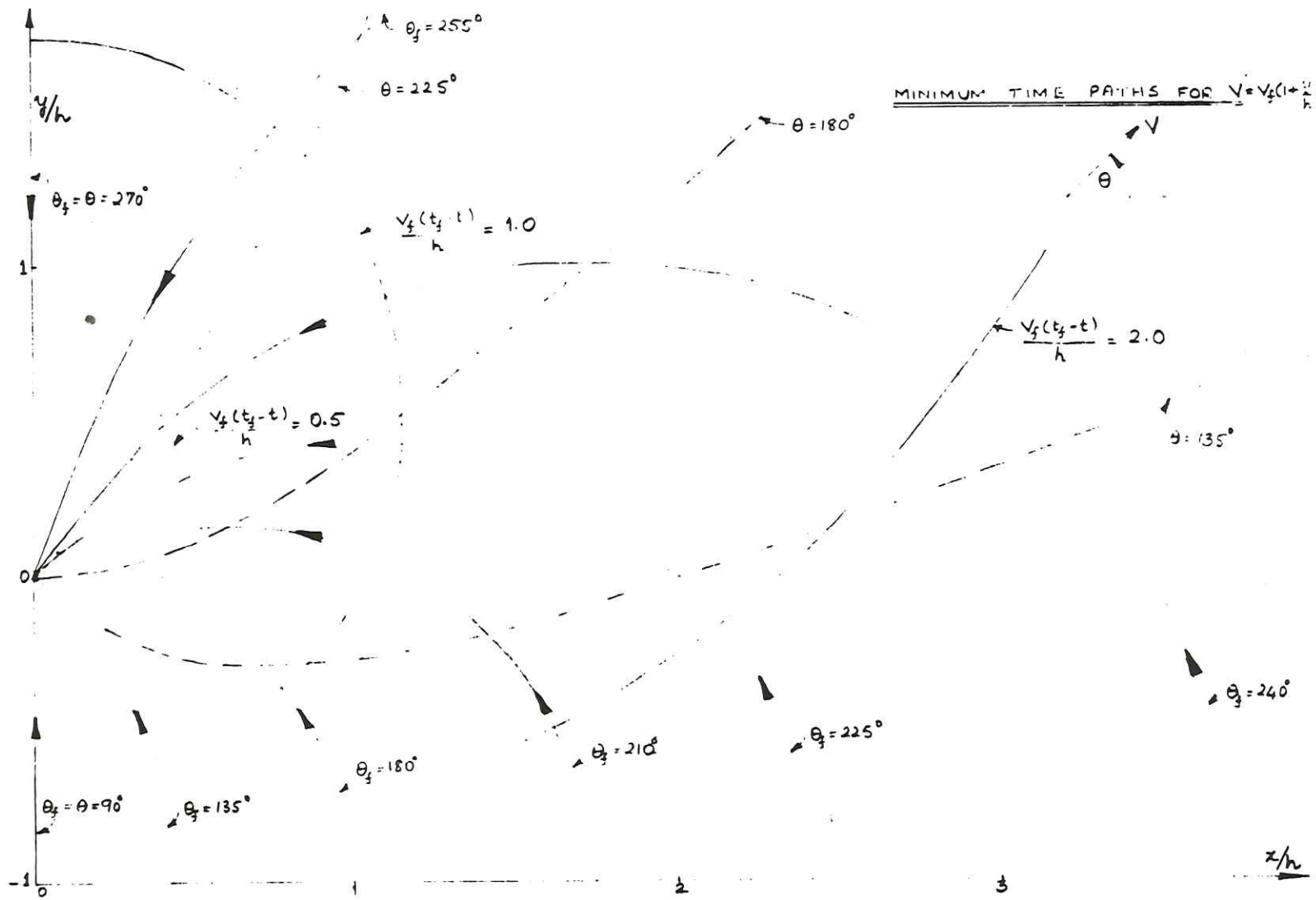
$$x^2 + \left[y - h\left\{\cosh\left(\frac{V_f\tau}{h}\right) - 1\right\}\right]^2 = \frac{h^2}{2}\left[\cosh\left(\frac{2V_f\tau}{h}\right) - 1\right] \qquad (14)$$

Eq. (14) are the isochrones, which are circles centred on the positive y-axis. In summary,

Minimum time paths $\Rightarrow$ eq.(7) or eq(11)

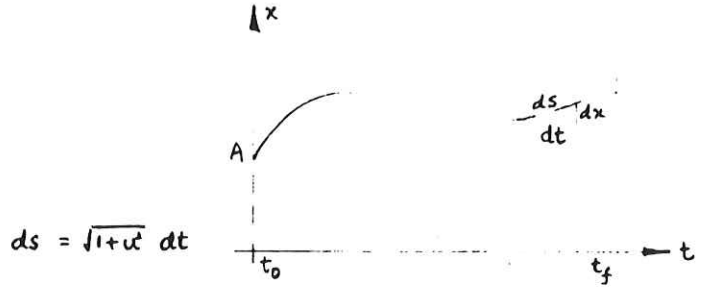
Contours of Const. time-to-go $\Rightarrow$ eq.(14)

Contours of Const. θ $\Rightarrow$ eq.(8); these are sketched on the accompanying diagram.


MINIMUM TIME PATHS FOR $V = V_f(1 + \frac{y}{h})$
$\theta_f = 255°$
$\theta = 225°$
$\theta = 180°$
$\theta_f = \theta = 270°$
$\frac{V_f(t_f - t)}{h} = 1.0$
$\frac{V_f(t_f - t)}{h} = 2.0$
$\frac{V_f(t_f - t)}{h} = 0.5$
$\theta = 135°$
$\theta_f = 240°$
$\theta_f = 225°$
$\theta_f = 210°$
$\theta_f = 180°$
$\theta_f = \theta = 90°$
$\theta_f = 135°$
V
θ
y/h
x/h
1
0
-1
0
1
2
3

PROBLEM 4.2.1.

This problem is somewhat related to problem 2.7.14. We formulate the "minimum distance in the (x,t) plane" problem as follows:



$$\min_{u} J = \int_{t_0}^{t_f} \sqrt{1+u^2}\, dt \quad \text{subject to} \quad \dot{x} = u \,. \tag{1}$$

Let $J^0(x,t)$ be the optimal return function, starting from (x,t). If u_0 is the optimal control, (from t to t_f),

$$J^0(x,t) = \int_{t}^{t_f} \sqrt{1+u_0^2}\, dt \tag{2}$$

$$= \int_{t}^{t+\Delta t} \sqrt{1+u_0^2}\, dt + \int_{t+\Delta t}^{t_f} \sqrt{1+u_0^2}\, dt$$

Clearly for some u different from u_0 for $t \in [t, t+\Delta t]$,

$$J'(x,t) \triangleq \int_{t}^{t+\Delta t} \sqrt{1+u^2}\, dt + \int_{t+\Delta t}^{t_f} \sqrt{1+u_0^2}\, dt \geq J^0(x,t)$$

Note that x_0 at $t = t_0$ is specified, t_f is fixed, and x_f is free.

Eq. (4.2.11) for this problem becomes,

$$\min_{u} \left[\sqrt{1+u^2} + \frac{\partial J^0}{\partial x} u \right] = -\frac{\partial J^0}{\partial t}, \quad \text{with} \tag{3}$$

the boundary condition. $J^0(x(t_f), t_f) = 0$. (4)

Note that the value of u that minimizes the bracketed

expression in eq. (3) is finite only if $|\frac{\partial J^0}{\partial x}| < 1$

If $|\frac{\partial J^0}{\partial x}| > 1$ an unbounded u may be chosen to yield $-\infty$ as the minimum value of the bracketed quantity in (3). Assuming, $|\frac{\partial J^0}{\partial x}| < 1$, we can write minimizing condition for (3) as,

$$\frac{u}{\sqrt{1+u^2}} + \frac{\partial J^0}{\partial x} = 0 ,$$

$$\Rightarrow \qquad u = \frac{-\frac{\partial J^0}{\partial x}}{\sqrt{1-(\frac{\partial J^0}{\partial x})^2}} . \tag{5}$$

Using (5) the HJB eqn. (3) becomes,

$$0 = \frac{\partial J^0}{\partial t} + \left[1-\left(\frac{\partial J^0}{\partial x}\right)^2\right]^{1/2} . \tag{6}$$

The solution to (6) is,

$$J^0 = t_f - t \tag{7}$$

which clearly satisfies the boundary condition (4).

To obtain the optimal control function, $u(x,t)$ we use (7) in (5). Thus,

$$\frac{\partial J^0}{\partial x} \equiv 0 \quad \Rightarrow \quad u^0(x,t) \equiv 0 \tag{8}$$

Eq. (7) gives the optimal return function, while (8) is the optimal feed-back control. Hence, the optimal (extremal) curve joining (x_0, t_0), $t_0 < t_f$, with the line $t = t_f$, which minimizes

the distance in the plane between end points, is indeed a straight line of zero slope [$u^0 = 0$] and the distance in the plane is $t_f - t_0$.

Now consider the end point fixed at (x_f, t_f). The optimal return must still satisfy the HJB PDE ③, but the boundary condition is now different.

We might try the following (obvious) solution:

$$J^0(x,t) = \left[(x_f - x)^2 + (t_f - t)^2 \right]^{1/2} \tag{9}$$

The verification of (9) as being the true solution is as follows:

$$\frac{\partial J^0}{\partial x} = \frac{-(x_f - x)}{J^0(x,t)} \tag{10}$$

$$\frac{\partial J^0}{\partial t} = \frac{-(t_f - t)}{J^0(x,t)} \tag{11}$$

Substituting (10) and (11) in (6), we see that (9) is the true solution to (3). (The boundary condition thus far is implicit and will be checked below.)

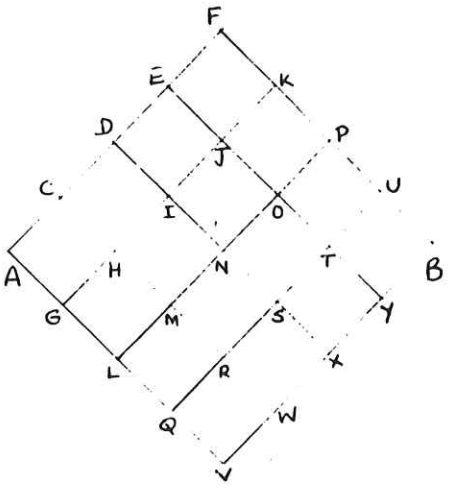
$$\text{Eq. (9)} \Rightarrow \quad u(x,t) = \frac{x_f - x}{t_f - t} = \frac{dx}{dt} \tag{12}$$

The differential eq. (12) is easily integrated:

$$x - x_f = c(t - t_f), \quad c = \text{const.} \tag{13}$$

Eq. (13) is a family of straight lines passing through (x_f, t_f), thus satisfying the boundary condition. Our particular solution is the one passing through (x_0, t_0).

PROBLEMS 4.2.2 and 4.2.3.



(i) T_{min} = 65

Path : ACHIJOPUB

(ii) T_{max} = 93

Path : AGHINOTYB

PROBLEM 4.2.4

Refer Fig. 4.2.5 in text. $\Rightarrow$ T_{min} = 406

Path : ACHLM

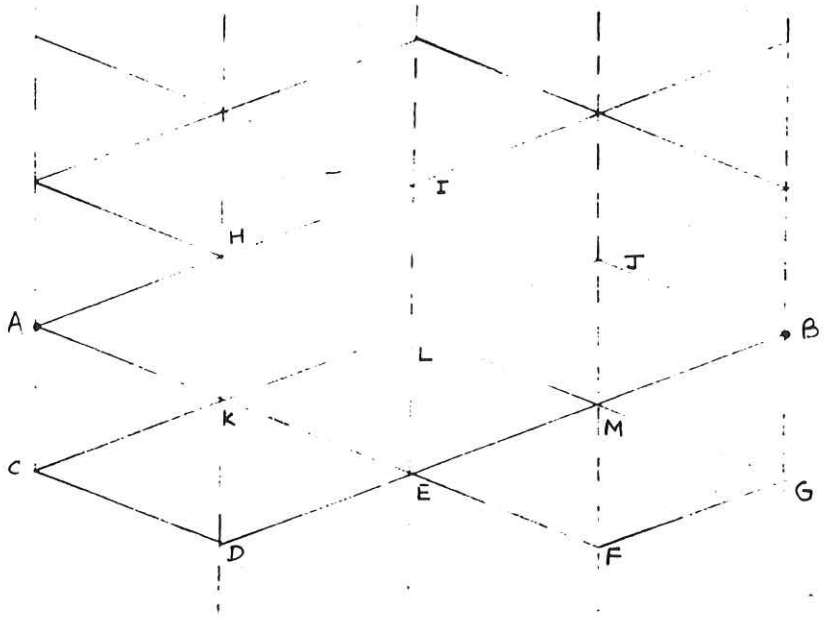
PROBLEM 4.2.5

Assuming the cost for the path ⑨ - ⑦ to be zero, we get, T_{min} = 21 along the path,

⑫ ⑩ ⑨ ⑦ ⑤ ② ①. (refer Fig. 4.2.6 in text)

PROBLEM 4.2.6.

Again this problem can be solved as the previous 4 problems. (Refer to Figure below.)



The purpose of this sketch is to identify some points of interest – for details refer Fig 4·2·7 in text.

Minimum time paths for (a), (b), (e) are all the same ; $\Rightarrow$ AKEMB

Minimum time path for (c) $\Rightarrow$ CKEMB

Minimum time path for (d) $\Rightarrow$ AKE

The costs (minimum time paths as above) are,

(a) $\Rightarrow$ 11

(b) $\Rightarrow$ 11

(c) $\Rightarrow$ 7

(d) $\Rightarrow$ 7

(e) $\Rightarrow$ 12

For (e) we cannot write u^0 in the form,

$$u^0 = k(x,t) .$$

This is because x,t alone do not determine the cost from x,t (unless x,t happens to be the initial starting point).

(f) The various analogs are:

t : level

$x(t)$: vertical position at level t (there are 8 possible positions, only 4 being allowed at any position). These positions may be numbered 1 through 8 starting from "top".

$u(t)$: decision to go up or down at any level

$f(x,u,t)$: relationship between present x, u and next grid point (taking $u = \pm 1$ to mean up or down we have $f = x+u$, ofcourse u is constrained to be ± 1 at any x,t)

$\phi(x(t_f), t_f)$: terminal cost at level 4. For eg,

$\phi[x(t_f), t_f] = 0$ for $t_f < 4$, $\phi(1,4) = 1$,

$\phi(3,4) = 3$, $\phi(5,4) = 2$, $\phi(7,4) = 4$.

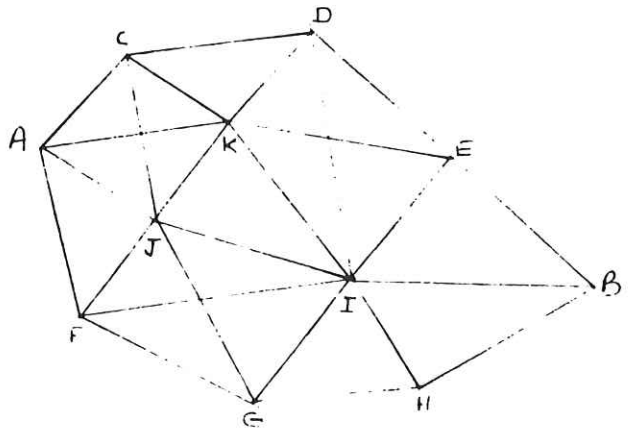
$L(x,u,t)$: cost of travel to the right from (x,t) along u.

$\psi(x(t_f), t_f)$: terminal manifold . For (a), (c) and (d) ψ is level 4 (states 1, 3, 5, 7), for (b) there is no terminal manifold but a point B $\equiv (5, 4)$, for (d) $\psi \equiv S$

$u^0(x,t)$: optimal decision (up or down) at any point

$J^0(x,t)$ cost to go from x, t to ψ using u^0. ie the optimal cost associated with each grid point.

PROBLEM 4.2.7



$T_{min} = 18$,

path : AKDIEB

PROBLEM 4.3.1

In order to use the results of problem 2.7.6, we need to make only the following substitutions:

$$y \rightarrow y - y_f$$

$$x \rightarrow -x \quad \text{or} \quad \theta \rightarrow \pi - \gamma$$

Then,

$$\left.\begin{aligned} V &= \sqrt{V_f^2 + 2g(y - y_f)} \\ \dot{x} &= -V(y)\cos\gamma \\ \dot{y} &= V(y)\sin\gamma \end{aligned}\right\} \quad (1)$$

Since $y(t_f)$ is not specified, $\lambda_y(t_f) = 0$

Also

$$\left.\begin{aligned} \tan\gamma &= -\frac{\lambda_y}{\lambda_x}, \quad \lambda_x = \text{const} \\ \Rightarrow \quad \gamma_f &= 0 \end{aligned}\right\} \quad (2)$$

Moreover,

$$\dot{\gamma} = -\frac{g\cos\gamma}{V} = g\lambda_x = \text{const} = -\frac{g}{V_f} \quad (3)$$

$$\Rightarrow \quad \gamma = \cos^{-1}\frac{V}{V_f}, \quad \text{and} \quad (4)$$

$$V_f\, d\gamma = -g\, dt \quad (5)$$

or,

$$\frac{g}{V}(t_f - t) = \frac{\pi/2 - \sin^{-1}(V/V_f)}{(V/V_f)} \quad (6)$$

Eq. (1) $\Rightarrow$

$$\frac{dy}{d\gamma} \equiv \frac{\dot{y}}{\dot{\gamma}} = -\frac{V^2}{g}\tan\gamma = -\frac{V_f^2\cos^2\gamma}{g}\tan\gamma$$

Integrating,

$$y_f - y = \int_{\gamma}^{0} -\frac{V_f^2}{g}\frac{1}{2}\sin 2\gamma'\, d\gamma'$$

$$\Rightarrow \quad \frac{y_f - y}{x} = \frac{V^2}{2gx}\left[\left(\frac{V_f}{V}\right)^2 - 1\right] \quad (7)$$

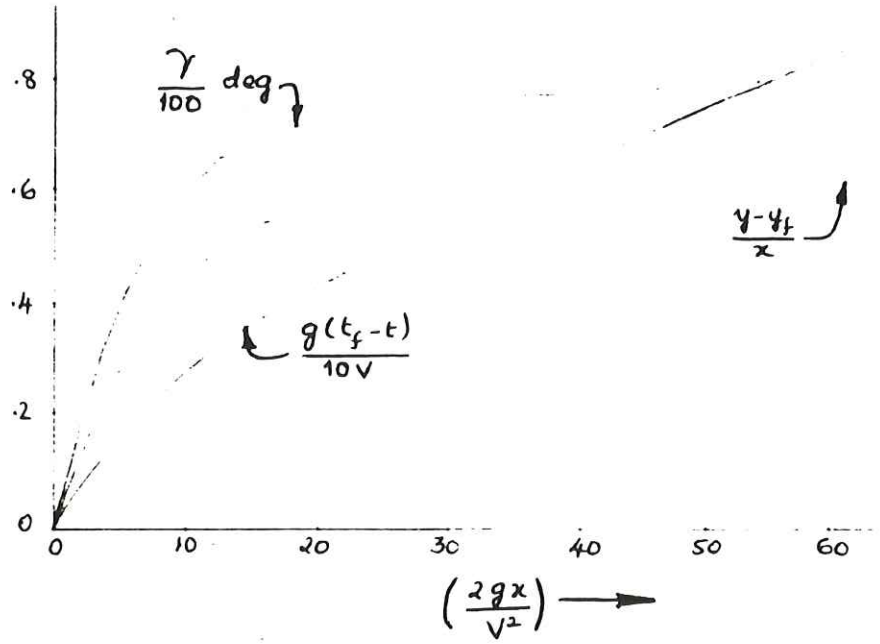
Similarly, $$\frac{dx}{d\gamma} = \frac{\dot{x}}{\dot{\gamma}} = \frac{V^2}{g} = \frac{V_f^2}{g}\cos^2\gamma$$

$$\Rightarrow \quad -x + 0 = \frac{V_f^2}{2g}\int_{\gamma}^{0}(1+\cos 2\gamma')\,d\gamma'$$

$$\text{or} \quad \frac{2gx}{V^2} = \left(\frac{V_f}{V}\right)^2\left[\gamma + \sin\gamma\cos\gamma\right]$$

$$\Rightarrow \quad \frac{2gx}{V^2} = \left(\frac{V_f}{V}\right)^2\left[\frac{\pi}{2} - \sin^{-1}\frac{V}{V_f}\right] + \frac{V_f}{V}\sqrt{1-\left(\frac{V}{V_f}\right)^2} \qquad (8)$$

The relevant graph is plotted below:



PROBLEM 4.3.2

We use the results from problem 2.7.6 and write the following equations:

(Note the difference between PROBLEM 2.7.6 and 4.3.2, also read the comments in the beginning of the previous problem)

$$V = (V_c^2 + 2gy)^{1/2} \tag{1}$$

$$x = x_c + \frac{V_c^2}{4g\cos^2\gamma_c}\left[2(\gamma-\gamma_c) + \sin 2\gamma - \sin 2\gamma_c\right] \tag{2}$$

$$y = y_c - \frac{V_c^2}{4g\cos^2\gamma_c}\left[\cos 2\gamma - \cos 2\gamma_c\right] \tag{3}$$

Note that eqns. (2) & (3) are nothing but eqns. (10) & (11) of Problem 2.7.6, incorporating the initial conditions x_0, y_0 and taking account of the correspondences $\theta \to \pi - \gamma$ etc. Since we are interested in a feedback solution let $x \to x_f = 0$, $x_0 \to x$ etc. to get.

$$x = \frac{-V^2}{4g\cos^2\gamma}\left[2(\gamma_f-\gamma) + \sin 2\gamma_f - \sin 2\gamma\right] \tag{4}$$

$$y = \frac{V^2}{4g\cos^2\gamma}\left[\cos 2\gamma_f - \cos 2\gamma\right] \tag{5}$$

$$\Rightarrow \frac{-x}{y} = \frac{2(\gamma_f-\gamma) + \sin 2\gamma_f - \sin 2\gamma}{\cos 2\gamma_f - \cos 2\gamma} \tag{6}$$

$$\frac{2gx}{V^2} = \frac{2(\gamma_f-\gamma) + \sin 2\gamma_f - \sin 2\gamma}{-2\cos^2\gamma} \tag{7}$$

Now since $\dot\gamma$ is a constant (Problem 4.3.1, eq.(3))

$$t_f - t = \frac{\gamma_f - \gamma}{\dot\gamma} = \frac{V(\gamma-\gamma_f)}{g\cos\gamma}$$

$$\Rightarrow \frac{g(t_f-t)}{V} = \frac{\gamma-\gamma_f}{\cos\gamma} \tag{8}$$

CHAPTER 5.

LINEAR SYSTEMS WITH QUADRATIC CRITERIA : LINEAR FEEDBACK

PROBLEM 5.2.1.

Minimize $$J = \tfrac{1}{2} x^T S_f x \Big|_{t_f} + \tfrac{1}{2} \int_{t_0}^{t_f} (x^T A x + u^T B u)\, dt \,, \qquad (1)$$

subject to, $$\dot{x} = Fx + Gu \,, \quad x(t_0), t_f \text{ given} \qquad (2)$$

Let, $u = kx$, where $k \equiv k(t)$ is the matrix of control variables.

$$\Rightarrow \quad J = \tfrac{1}{2} x^T S_f x \Big|_{t_f} + \tfrac{1}{2} \int_{t_0}^{t_f} x^T (A + K^T B K) x \, dt \,, \qquad (3)$$

$$\dot{x} = (F + GK) x \,. \qquad (4)$$

As usual,

$$\left.\begin{aligned} H &= \tfrac{1}{2} x^T (A + K^T B K) x + \lambda^T (F + GK) x \\ \dot{\lambda}^T &= - x^T (A + K^T B K) - \lambda^T (F + GK) \\ \lambda(t_f) &= S_f x(t_f) \\ 0 = \frac{\partial H}{\partial K} &= \tfrac{1}{2} \frac{\partial}{\partial K} (x^T K^T B K x) + \frac{\partial}{\partial K} (\lambda^T G K x) \end{aligned}\right\} \qquad (5)$$

Now we use the "hint" to differentiate wrt a matrix.

(i) $$\frac{\partial}{\partial K} (x^T K^T B K x) = \frac{\partial}{\partial K} \left(Tr [x^T K^T B K x] \right) = \frac{\partial}{\partial K} \left(Tr [x x^T K^T B K] \right)$$
$$= 2 x x^T K^T B \qquad (6)$$

(ii) $$\frac{\partial}{\partial K} (\lambda^T G K x) = \frac{\partial}{\partial K} \left(Tr [x \lambda^T G K] \right) = x \lambda^T G \qquad (7)$$

Using eqs (6) & (7) in (5), the optimality condition becomes,

$$x x^T K^T B + x \lambda^T G = 0$$

$$\Rightarrow \quad x (x^T K^T B + \lambda^T G) = 0$$

$$\Rightarrow \quad u \equiv Kx = - B^{-1} G^T \lambda \qquad (8)$$

Eqns. (4), (5) & (8) yield the TPBVP,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} F & -GB^{-1}G^T \\ -A & -F^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \; ; \; x(t_0) \text{ given} \qquad (9)$$

$$\lambda(t_f) = S_f x(t_f). \qquad (10)$$

This can be solved by "Sweep method" as in text.

PROBLEM 5.2.2.

Minimize $J = \frac{1}{2} c x_f^2 + \frac{1}{2} \int_{t_0}^{t_f} u^2 \, dt$,

subject to $\dot{x} = -ax + bu \; ; \; x(t_0)$ given

As usual, $H = \frac{1}{2} u^2 + \lambda(-ax + bu)$

$$\left.\begin{aligned} \dot{\lambda} &= a\lambda \; ; \; \lambda_f = c x_f \\ 0 = H_u &= u + b\lambda \end{aligned}\right\} \qquad (1)$$

Eq. (1) $\Rightarrow \lambda(t) = c x_f e^{-a(t_f - t)}$, (2)

$$u(t) = -bc x_f e^{-a(t_f - t)}. \qquad (3)$$

Hence, $\dot{x} = -ax - b^2 c x_f e^{-a(t_f - t)}$

$$\Rightarrow x(t) = x_0 e^{-a(t-t_0)} - \frac{b^2 c}{2a} x_f e^{-a(t_f - t_0)} \left[e^{a(t-t_0)} - e^{-a(t-t_0)} \right] \qquad (4)$$

$$\Rightarrow x_f \equiv x(t_f) = \frac{x_0 e^{-a(t_f - t_0)}}{1 + \frac{b^2 c}{2a} \left[1 - e^{-2a(t_f - t_0)} \right]} \qquad (5)$$

Using (5) in (3) we get the sampled data feedback control law :

$$u(t,t_0) = -b\,\frac{\exp[-a(t_f-t_0)-a(t_f-t)]}{1/c + b^2/2a\,(1-\exp(-2a(t_f-t_0)))}\,x(t_0) \qquad (6)$$

Replacing t_0 by t, the continuous feedback law is,

$$u(t) = -b\,\frac{\exp[-2a(t_f-t)]}{1/c + \frac{b^2}{2a}\{1-\exp[-2a(t_f-t)]\}}\,x(t) \qquad (7)$$

$$\text{Also, } x(t) = \frac{e^{a(t_0-t)}\left[1+\frac{b^2c}{2a}(1-e^{-2a(t_f-t)})\right]}{1+\frac{b^2c}{2a}(1-e^{-2a(t_f-t_0)})}\,x(t_0)\,, \qquad (8)$$

$$\lambda(t) = \frac{c\,e^{-a(t_f-t_0)}\,e^{a(t-t_f)}}{1+\frac{b^2c}{2a}(1-e^{-2a(t_f-t_0)})}\,x(t_0)\,. \qquad (9)$$

$$\text{Hence } S(t) = \frac{\lambda(t)}{x(t)} = \frac{c\,e^{-2a(t_f-t)}}{1+\frac{b^2c}{2a}(1-e^{-2a(t_f-t)})}\,. \qquad (10)$$

Eq.(5) $\Longrightarrow$ $x(t_f) \to 0$ as $c \to \infty$.

PROBLEM 5.2.3

This problem will be solved without many comments.

$$\dot{x}_1 = x_2\,, \quad x_1(t_0) \text{ given} \qquad (1)$$

$$\dot{x}_2 = -\omega^2 x_1 + u\,, \quad x_2(t_0) \text{ given} \qquad (2)$$

$$J = \tfrac{1}{2}\,c\,x_{1f}^2 + \tfrac{1}{2}\int_{t_0}^{t_f} u^2\,dt \qquad (3)$$

Hamiltonian : $H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2(-\omega^2 x_1 + u)$

Optimality : $H_u = 0 \Rightarrow u = -\lambda_2$ (4)

Adjoint equations:

$$\left.\begin{aligned} \dot{\lambda}_1 &= \omega^2 \lambda_2 \quad , \lambda_1(t_f) = c\, x_1(t_f) \\ \dot{\lambda}_2 &= -\lambda_1 \quad , \lambda_2(t_f) = 0 \end{aligned}\right\} \tag{5}$$

Eq. (5) $\Longrightarrow$ $\ddot{\lambda}_1(t) = -\omega^2 \lambda_1$

or, $$\lambda_1(t) = c\, x_1(t_f) \cos \omega(t_f - t) \tag{6}$$

and $$\lambda_2(t) = \frac{c\, x_1(t_f)}{\omega} \sin \omega(t_f - t) \equiv -u \tag{7}$$

Using (6) & (7) the state eqns are rewritten:

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{c\, x_1(t_f)}{\omega} \sin \omega(t_f - t) \quad , \dot{x}_1(t_0) = x_2(t_0), \; x_1(t_0) \text{ given} \tag{8}$$

Now, $\sin \omega(t - t_f) = \sin \omega(t - t_0) \cos \omega(t_0 - t_f) + \cos \omega(t - t_0) \sin \omega(t_0 - t_f)$

so that, eq. (8) has the solution,

Complementary function : $A \cos \omega(t - t_0) + B \sin \omega(t - t_0)$

Particular integral : $\frac{c\, x_1(t_f)}{\omega} \left[\frac{t - t_0}{2\omega} \cos \omega(t_0 - t_f) \cos \omega(t - t_0) - \frac{t - t_0}{2\omega} \sin \omega(t_0 - t_f) \sin \omega(t - t_0) \right]$.

Evaluating A and B is routine and we get finally,

$$x_1(t) = x_1(t_0) \cos \omega(t - t_0) + \frac{x_2(t_0)}{\omega} \sin \omega(t - t_0) + \frac{c\, x_1(t_f)}{2\omega^2} \left[\frac{\cos \omega(t_f - t_0) \sin \omega(t - t_0)}{\omega} - (t - t_0) \cos \omega(t_f - t) \right] \tag{9}$$

$$\Longrightarrow x_1(t_f) = \frac{x_1(t_0) \cos \omega(t_f - t_0) + \frac{x_2(t_0)}{\omega} \sin \omega(t_f - t_0)}{1 + \frac{c}{4\omega^3} \left[2\omega(t_f - t_0) - \sin 2\omega(t_f - t_0) \right]} \tag{10}$$

Clearly $x_1(t_f) \to 0$ as $c \to \infty$.

Using (10) in (7), the sampled data feedback control law is :

$$u(t,t_0) = -\frac{4w^2 x_1(t_0)\cos w(t_f-t_0) + 4w\, x_2(t_0)\sin w(t_f-t_0)}{4\frac{w^3}{c} + 2w(t_f-t_0) - \sin 2w(t_f-t_0)} \sin w(t_f-t)$$

Letting $t_0 = t$, we get the continuous feedback law:

$$u(t) = -\frac{4w^2 x_1(t)\cos w(t_f-t) + 4w\, x_2(t)\sin w(t_f-t)}{4\frac{w^3}{c} + 2w(t_f-t) - \sin 2w(t_f-t)} \sin w(t_f-t)$$

As $t \to t_f$, clearly $u(t) \to \infty$

(Note: $\lim_{t\to t_f} \dfrac{\sin w(t_f-t)}{\frac{4w^3}{c} + 2w(t_f-t) - \sin 2w(t_f-t)}$, $\left[\frac{4w^3}{c} \to 0 \text{ as } c \to \infty\right]$

$$= \frac{1}{\lim_{t\to t_f}\left[\frac{2w(t_f-t)}{\sin w(t_f-t)} - \frac{\sin 2w(t_f-t)}{\sin w(t_f-t)}\right]}$$

$$= \frac{1}{2-2} \longrightarrow \infty \quad)$$

PROBLEM 5.2.4

$$H = \frac{1}{2}\left[x^T A x + x^T N u + u^T N^T x + u^T B u\right] + \lambda^T(Fx+Gu)$$

$$0 = H_u \Rightarrow \tfrac{1}{2}x^T N + \tfrac{1}{2}x^T N + u^T B + \lambda^T G = 0$$

Let $\lambda(t) = S(t)\,x(t)$, so that,

$$u(t) = -B^{-1}(N^T + G^T S)\,x(t) \quad . \tag{1}$$

Now, $\dot\lambda = -H_x^T = -F^T S x - Ax + NB^{-1}(N^T+G^T S)\,x$ (2)

But $\dot\lambda = \dot S x + S\dot x = \dot S x + S\left(Fx - GB^{-1}(N^T+G^T S)x\right)$ (3)

Eqns. (2) & (3) $\Rightarrow \dot S = -SF - F^T S + (SG+N)B^{-1}(N^T+G^T S) - A$, (4)

with $S(t_f) = S_f$ (5)

The equivalent problem has,

$$H = \frac{1}{2} x^T (A - NB^{-1}N^T) x + \frac{1}{2} u^T B u + \lambda^T (F - GB^{-1}N^T) x + \lambda^T G u$$

$$\dot{\lambda} = -(A - NB^{-1}N^T) x - (F^T - NB^{-1}G^T) S x$$

$$= \dot{S} x + S (F - GB^{-1}N^T) x + SG\left[-B^{-1}G^T S x\right]$$

$$\Rightarrow \dot{S} = -SF - F^T S - A + (SG + N) B^{-1} (N^T + G^T S)$$

Which is the same equation as before (Note that the expressions for $u(t)$ are different, but the state equations — for optimal u — are same.)

PROBLEM 5.2.5.

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} F & -GB^{-1}G^T \\ -A & -F^T \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} x \\ \lambda \end{bmatrix}}_{z}$$

$$\Rightarrow \dot{z} = P z \qquad (1)$$

Let the transition matrix associated with P, be $\phi(t, t_0)$, so that

$$\frac{d}{dt}\phi(t, t_0) = P\,\phi(t, t_0) \qquad (2)$$

Now,

$$P = \begin{bmatrix} F & -GB^{-1}G^T \\ -A & -F^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \underbrace{\begin{bmatrix} A & F^T \\ F & -GB^{-1}G^T \end{bmatrix}}_{M}$$

$$\Rightarrow P = J M, \text{ where } M \text{ is symmetric.} \qquad (3)$$

Hence, $\frac{d}{dt}\phi(t, t_0) = JM\,\phi(t, t_0)$. (4)

It is easy to verify that, $J^{-1} = J^T = -J$ (5)

Consider, $\frac{d}{dt}\phi^T(t_0,t) = -P^T\phi^T(t_0,t)$

$= -MJ^T\phi^T(t_0,t)$

$\Rightarrow \frac{d}{dt}\left[J^T\phi^T(t_0,t)J\right] = -J^TMJ^T\phi^T(t_0,t)J$

$= JM\left[J^T\phi^T(t_0,t)J\right]$ (6)

Thus $J^T\phi^T(t_0,t)J$ satisfies the same differential equation (4) as $\phi(t,t_0)$. Moreover,

$J^T\phi^T(t_0,t_0)J = J^TJ = I$.

Hence, $J^T\phi^T(t_0,t)J \equiv \phi(t,t_0)$ (7)

$\Rightarrow \phi^T(t_0,t)J\phi^{-1}(t,t_0) = J$

Or $\phi^T(t_0,t)J\phi(t_0,t) = J$ (8)

Now, eq.(7) $\Longrightarrow \phi^T(t_0,t) = J\phi(t,t_0)J^T$

or, $\phi(t_0,t) = J\phi^T(t,t_0)J^T$

Hence,

$$\phi^{-1}(t,t_0) \equiv \phi(t_0,t) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} \phi_{xx}^T & \phi_{\lambda x}^T \\ \phi_{x\lambda}^T & \phi_{\lambda\lambda}^T \end{bmatrix}\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{\lambda\lambda}^T(t,t_0) & -\phi_{x\lambda}^T(t,t_0) \\ -\phi_{\lambda x}^T(t,t_0) & \phi_{xx}^T(t,t_0) \end{bmatrix} \quad (9)$$

PROBLEM 5.2.6

$$\dot{x} = u$$

$$J = \frac{1}{2}\frac{[x(0)]^2}{P} + \frac{1}{2}\int_0^{t_f}\left[\frac{u^2}{q} + \frac{(z-x)^2}{r}\right]dt$$

$$H = \frac{1}{2}\frac{u^2}{q} + \frac{1}{2}\frac{(z-x)^2}{r} + \lambda u$$

$$0 = H_u = \frac{u}{q} + \lambda \quad \Longrightarrow \quad u = -\lambda q$$

$$\dot{\lambda} = \frac{z-x}{r}, \qquad \lambda(0) = \frac{x(0)}{P}, \qquad \lambda(t_f) = 0$$

We use sweep method ;

(look at eq. 13·3·5 on p. 396 of text)

let $\lambda(t) = s(t)\,x(t) + a(t)$; $s(0) = \frac{1}{P}$; $a(0) = 0$ (1)

$$\Rightarrow \dot{\lambda} = \frac{z-x}{r} = \dot{s}x + s\dot{x} + \dot{a}$$

$$= \dot{s}x + s\left[-q(sx+a)\right] + \dot{a}$$

$$\Longrightarrow \quad \dot{s} = qs^2 - \frac{1}{r}, \quad s(0) = \frac{1}{P} \qquad (2)$$

$$\dot{a} = aqs + \frac{z}{r}, \quad a(0) = 0 \qquad (3)$$

also, $\dot{x} = -(sx+a)q, \quad x(0) = 0 \qquad (4)$

Once the measurement scheme is known (ie. P, q, r are known) eq. (2) may be integrated and $s(t)$ values stored, for $0 \le t \le t_f$. As the measurements $z(t)$ arrive, eq. (3) may be integrated to yield $a(t)$ and hence $x(t)$ obtained from (4).

Note : The above method should be compared with the "smoothing" derivation (in 13.3) in the text.

PROBLEM 5·3·1 (METHOD 1)

Consider the transformation,

$$u = v + Kx \tag{1}$$

where v is a new control variable and K is a matrix which will be chosen by us so as to eliminate the "cross terms" in J.

Let, $\Lambda = x^T A x + 2 x^T N u + u^T B u$, (2)

$$(1) \Rightarrow \Lambda = x^T A x + 2 x^T N (v + Kx) + v^T B v + 2 v^T B K x + x^T K^T B K x$$

$$= x^T (A + 2NK + K^T B K) + 2 v^T (BK + N^T) x$$

We choose, $K = -B^{-1} N^T$, so that

$$\Lambda = x^T (A - N B^{-1} N^T) x + v^T B v \tag{3}$$

(Refer to problem 5·2·4 now!)

Our problem now reduces to:

$$\text{Min. } J = \frac{1}{2} \int_{t_0}^{t_f} \left[x^T (A - N B^{-1} N^T) x + v^T B v \right] dt$$

$$\text{s.t. } \dot{x} = (F - G B^{-1} N^T) x + G v \quad ; \; x(t_0) \text{ given}$$

$$\psi = M x(t_f)$$

Section 5·3 now applies, with the following changes:

$A \to A - N B^{-1} N^T$, $F \to F - G B^{-1} N^T$, $\psi = M x(t_f)$.

We get the required equations for $\dot{S}, \dot{R}, \dot{Q}$ with little difficulty. Note that, $R_{ij}(t_f) = \left(\frac{\partial \psi_i}{\partial x_i} \right)_{t=t_f} = M_{ij}^T$

$$\Rightarrow \quad R(t_f) = M^T$$

To develop the expression for J_{min}, we eliminate V in eq. (3), to get,

$$\Lambda = x^T(A - N\bar{B}^1N^T)x + x^TSG\bar{B}^1G^TSx + 2x^TSG\bar{B}^1G^TR\bar{Q}^1(\psi - \hat{\psi}) + \underbrace{(\psi - \hat{\psi})^T \bar{Q}_{t_0}^{-1} R^T G\bar{B}^1G^T R\bar{Q}_{t_0}^{-1}(\psi - \hat{\psi})}_{\Lambda_2}$$

Let $\Lambda \equiv \Lambda_1 + \Lambda_2$

It is shown in detail (in METHOD 2) that $\int_{t_0}^{t_f} \Lambda_2 \, dt$ yields the second term in J_{min}.

Thus we need to show, $\int_{t_0}^{t_f} \Lambda_1 \, dt = x^TSx\Big|_{t=t_0}$

$$\begin{aligned}
\Lambda_1 &\equiv x^T(A - N\bar{B}^1N^T + SG\bar{B}^1G^TS)x - 2x^TSG\left[V + \bar{B}^1G^TSx\right] \\
&= x^T(A - N\bar{B}^1N^T - SG\bar{B}^1G^TS)x - 2x^TSGV \\
&= x^T(A - N\bar{B}^1N^T - SG\bar{B}^1G^TS)x - 2x^TS(\dot{x} - Fx + G\bar{B}^1N^Tx) \\
&= x^T(A - N\bar{B}^1N^T - SG\bar{B}^1G^TS + 2SF - 2SG\bar{B}^1N^T)x - 2x^TS\dot{x} \\
&= x^T(A - N\bar{B}^1N^T - SG\bar{B}^1G^TS + SF + F^TS - SG\bar{B}^1N^T - N\bar{B}^1G^TS)x - 2x^TS\dot{x} \\
&= -2x^TS\dot{x} - x^T\dot{S}x \\
&= \frac{d}{dt}(-x^TSx)
\end{aligned}$$

Hence J_{min} = required expression.

In what follows, a more straightforward method is expounded, and some more details are worked out, for this same problem.

PROBLEM 5.3.1 (METHOD 2)

$$\dot{x} = F(t)\,x + G(t)\,u \quad , \quad x(t_0) \text{ given}$$

$$Mx(t_f) = \psi \quad , (\text{a given } q \text{ vector})$$

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x^T\ u^T) \begin{pmatrix} A(t) & N(t) \\ N^T(t) & B(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt$$

$$H = \frac{1}{2}(x^TAx + x^TNu + u^TN^Tx + u^TBu) + \lambda^T(Fx+Gu)$$

Necessary conditions :

$$0 = H_u = x^TN + u^TB + \lambda^TG \Rightarrow u = -B^{-1}[G^T\lambda + N^Tx] \quad (1)$$

$$\dot{\lambda} = -H_x^T = -Ax - Nu - F^T\lambda \quad , \quad \lambda(t_f) = M^T\nu \quad (2)$$

Using sweep method, let

$$\lambda(t) = S(t)\,x(t) + R(t)\,\nu \quad , \quad S(t_f) = 0 \quad , \quad R(t_f) = M^T, \quad (3)$$

$$\psi(t) = R^T(t)\,x(t) + Q(t)\,\nu \quad , \quad Q(t_f) = 0 \quad (4)$$

Using (3) in (2) and substituting for $\dot{x}$,

$$\dot{S}x + S\left[Fx - GB^{-1}\{G^T(Sx+R\nu) + N^Tx\}\right] + \dot{R}\nu$$

$$= -Ax + NB^{-1}[G^T(Sx+R\nu) + N^Tx] - F^T(Sx+R\nu)$$

Equating coefficients of x, ν we get,

$$\dot{S} = -SF - F^TS - A + (SG+N)B^{-1}(G^TS+N^T); \quad S(t_f) = 0;$$

$$\dot{R} = -[F^T - (SG+N)B^{-1}G^T]R \quad ; \quad R(t_f) = M^T;$$

Differentiating eq. (4) w.r.t t,

$$0 = \dot{R}^Tx + R^T\left[Fx - GB^{-1}(G^T\{Sx+R\nu\} + N^Tx)\right] + \dot{Q}\nu$$

$$\Rightarrow \dot{Q} = R^TGB^{-1}G^TR \quad ; \quad Q(t_f) = 0.$$

Having integrated the $\dot{S}$, $\dot{R}$, $\dot{Q}$ equations backwards, we can solve for ν as,

$$\nu = [Q(t)]^{-1}\left[\psi - R^T(t)\,x(t)\right] \tag{5}$$

Eq. (1) $\Rightarrow$

$$\begin{aligned} u &= -B^{-1}\left[G^T\lambda + N^T x\right] \\ &= -B^{-1}\left[G^T\left\{Sx + R\left[Q(t)\right]^{-1}(\psi - R^T(t)x(t))\right\} + N^T x\right] \end{aligned}$$

or,

$$u = -B^{-1}\left[N^T + G^T(S - RQ^{-1}R^T)\right]x - B^{-1}G^T R\,Q^{-1}\psi \tag{6}$$

Alternatively,

$$\begin{aligned} \nu &= \left[Q(t_0)\right]^{-1}\left[\psi - R^T(t_0)\,x(t_0)\right] \\ &= \left[Q(t_0)\right]^{-1}\left[\psi - \hat{\psi}\right]. \end{aligned}$$

$$\Rightarrow \quad u = -B^{-1}(N^T + G^T S)x - B^{-1}G^T R\left[Q(t_0)\right]^{-1}(\psi - \hat{\psi}) \tag{7}$$

Consider,

$$J(x,t) = \frac{1}{2}\int_t^{t_f} \underbrace{(x^T A x + u^T B u + x^T N u + u^T N^T x)}_{Z}\,dt\ .$$

Adding and subtracting $\frac{1}{2}\int_t^{t_f}\frac{d}{dt}(x^T S x)\,dt$, to J,

$$J(x,t) = -\frac{1}{2}x^T S x\Big|_t^{t_f} + \frac{1}{2}\int_t^{t_f}\left\{Z + \frac{d}{dt}(x^T S x)\right\}dt \tag{8}$$

We make use of the following:

(i) $$S_f \equiv S(t_f) = 0 \tag{9}$$

(ii) $$\begin{aligned} \frac{d}{dt}(x^T S(t)\,x) &= \frac{\partial}{\partial t}\left[x^T S(t)\,x\right] + \frac{\partial}{\partial x}\left[x^T S x\right]\frac{dx}{dt} \\ &= x^T\dot{S}\,x + 2\,x^T S\,(Fx + Gu) \end{aligned} \tag{10}$$

Using (9), (10) and substituting for u and using the Riccati equation for $\dot{S}$, and the eqns. for ν, we arrive (after a lot of cancellations) at,

$$J_{min}(x,t) = \frac{1}{2} x^T S(t) x + \frac{1}{2} \int_t^{t_f} (\psi - \hat{\psi})^T Q^{-1}(t_0) R^T G B^{-1} G^T R \; Q^{-1}(t_0) (\psi - \hat{\psi}) \, dt$$

But, $(\psi - \hat{\psi})^T Q^{-1}(t_0) = \nu^T$ = const., and

$R^T G B^{-1} G^T R = \dot{Q}$, so that,

$$J_{min}(x,t) = \frac{1}{2} x^T S(t) x + \frac{1}{2} \int_t^{t_f} \nu^T \dot{Q} \nu \, dt$$

$$= \frac{1}{2} x^T S(t) x - \frac{1}{2} \nu^T Q(t) \nu$$

$$\Rightarrow J_{min}(x_0,t_0) = \frac{1}{2} \left[x^T S x + (\psi - \hat{\psi})(-Q^{-1})(\psi - \hat{\psi}) \right]_{t=t_0}$$

Note: $\nu = [Q(t_0)]^{-1} [\psi - R^T(t_0) x(t_0)]$

$= [Q(t)]^{-1} [\psi - R^T(t) x(t)]$.

PROBLEM 5.3.2.

With $A = 0$, the Riccati equation for S becomes,

$$\dot{S} = -SF - F^T S + SGB^{-1}G^T S \qquad (1)$$

We have the identity, $SS^{-1} = I$, which on differentiation yields,

$$\left(\frac{d}{dt} S\right) S^{-1} + S \frac{d}{dt}(S^{-1}) = 0$$

$$\Rightarrow \frac{d}{dt}(S^{-1}) = -S^{-1} \dot{S} S^{-1}$$

$$= FS^{-1} + S^{-1}F^T - GB^{-1}G^T . \qquad (2)$$

To show that the solution to (2) may be put as,

$$S^{-1}(t) = R^{-T}(t) \left[S_f^{-1} + \int_t^{t_f} R^T G B^{-1} G^T R \, dt \right] R^{-1}(t) . \qquad (3)$$

Clearly the boundary condition is satisfied by (3).

i.e $S^{-1}(t_f) = S_f^{-1}$.

Premultiplying by R^T and post multiplying by R,

$$\text{eq.(3)} \Rightarrow R^T S^{-1} R = S_f^{-1} + \int_t^{t_f} R^T G B^{-1} G^T R \, dt \quad (4)$$

Differentiating eq. (4) wrt time,

$$-R^T G B^{-1} G^T R = \dot{R}^T S^{-1} R + R^T \frac{d}{dt}(S^{-1}) R + R^T S^{-1} \dot{R}$$

$$\Rightarrow -G B^{-1} G^T = (R^T)^{-1} \dot{R}^T S^{-1} + \frac{d}{dt}(S^{-1}) + S^{-1} \dot{R} R^{-1}$$

But, eq. (5.3.49) in text $\Rightarrow$ $\dot{R} + F^T R = 0$

or, $\dot{R} R^{-1} = -F^T$

Hence, $-G B^{-1} G^T = -(F^T)^T S^{-1} + \frac{d}{dt}(S^{-1}) - S^{-1} F^T$

$$\Rightarrow \frac{d}{dt}(S^{-1}) = S^{-1} F^T + F S^{-1} - G B^{-1} G^T$$

Thus, the assumed solution satisfies the differential equation and boundary condition. Required result follows by uniqueness of the solution to the differential equation

PROBLEM 5.3.3

From problem 5.3.2 we have,

$$\frac{d}{dt}[S^{-1}] = F S^{-1} + S^{-1} F^T - G B^{-1} G^T \quad (1)$$

This is a matrix linear system and its solution (ref. P 453 of text) is,

$$S^{-1}(t) = \phi(t,t_f)\, S^{-1}(t_f)\, \phi^T(t,t_f) + \int_{t_f}^{t} \phi(t,\tau)\, G B^{-1} G^T \phi^T(t,\tau)\, d\tau$$

$$= \int_{t_f}^{t} \phi(t,\tau)\, G B^{-1} G^T \phi^T(t,\tau)\, d\tau \qquad (2)$$

Since the problem is a constant coefficient one, (F, G are constant matrices)

$$\phi(t,\tau) = e^{F(t-\tau)} = I + F(t-\tau) + \frac{F^2}{2!}(t-\tau)^2 + \cdots \qquad (3)$$

But by the Cayley - Hamilton theorem (which states that every matrix satisfies its characteristic eqn.) F^n, F^{n+1}, ... can all be expressed in terms of F^{n-1}, F^{n-2}, ..., I and as a result we have, [ref. "Sylvester formula" in literature]

$$\phi(t,\tau) = \sum_{k=0}^{n-1} a_k F^k, \quad \left[a_k \text{ are functions of } (t-\tau)\right] \qquad (4)$$

In order that $S^{-1}(t) > 0$ we must have,

$$\int_{t_f}^{t} \left\{ \sum_{k=0}^{n-1} a_k F^k \right\} G B^{-1} G^T \left\{ \sum_{k=0}^{n-1} a_k F^k \right\}^T d\tau > 0$$

But, $\left\{ \sum_{k=0}^{n-1} a_k F^k \right\} G = a_0 G + a_1 FG + \cdots + a_{n-1} F^{n-1} G \qquad (5)$

Assuming $F \Rightarrow (n \times n)$, $G \Rightarrow (n \times m)$, the matrix in (5) is $(n \times m)$ and can be written as :

$$= \left[G \,:\, FG \,:\, \cdots \,:\, F^{n-1} G \right] \begin{bmatrix} I_m\, a_0 \\ I_m\, a_1 \\ \vdots \\ I_m\, a_{n-1} \end{bmatrix}$$

Hence, $\left\{ \sum_{K=0}^{n-1} a_K F^K \right\} G = A R$ (6)

where, $A = [G \,!\, FG \,!\, \cdots \,!\, F^{n-1}G]$, (7)

and $R^T = [I_m a_0 \,!\, I_m a_1 \,!\, \cdots \,!\, I_m a_{n-1}]$ (8)

Note that I_m is $(m \times m)$ identity matrix and that A is $(n \times nm)$, R is $(nm \times n)$.

Thus, $\bar{S}^{-1}(t) = \int_{t_f}^{t} A R \bar{B}^{-1} R^T A^T \, d\tau$

$$= A \left[\int_{t_f}^{t} R \bar{B}^{-1} R^T \, d\tau \right] A^T \quad (9)$$

For $\bar{S}^{-1}(t) > 0$, it must be nonsingular,

$\Longrightarrow$ A must have maximal rank (n)

PROBLEM 5.3.4.

$$\hat{\psi} = R^T(t)\, x(t)$$

$$\Rightarrow \dot{\hat{\psi}} = R^T(t)\, \dot{x}(t) + \dot{R}^T(t)\, x(t)$$

$$= R^T(t) [F x + G u] + (-R^T F) x$$

or $\dot{\hat{\psi}} = R^T G u \quad ; \quad \hat{\psi}(t_0) = R^T(t_0)\, x(t_0)$

PROBLEM 5.3.5

The system, $\ddot{x} = x + u$ can be put in state variable form by defining, $x = x_1$, $\dot{x} = \dot{x}_1 = x_2$.

Our problem then reduces to :

minimize, $J = \frac{1}{2}\int_0^{t_f} u^2\,dt$ (1)

Subject to, $\dot{x}_1 = x_2$, $x_1(0) = x_0$, $x_1(t_f) = 0$ (2)

$\dot{x}_2 = -x_1 + u$, $x_2(0) = v_0$, $x_2(t_f) = 0$ (3)

Obviously the state equations $\Rightarrow$ $\ddot{x}_1 + x_1 = u$ (4)

Adjoint eqns: $\left.\begin{aligned}\dot{\lambda}_1 &= \lambda_2\\ \dot{\lambda}_2 &= -\lambda_1\end{aligned}\right\} \Rightarrow \ddot{\lambda}_2 + \lambda_2 = 0$ (5)

Optimality: $H_u = 0 \Rightarrow u = -\lambda_2$ (6)

Eq. (5) $\Rightarrow$ $\lambda_2 = -(A\sin t + B\cos t)$

Hence, $\ddot{x}_1 + x_1 = u = A\sin t + B\cos t$ (7)

$\Rightarrow x \equiv x_1 = C\cos t + D\sin t - \frac{1}{2}At\cos t + \frac{1}{2}Bt\sin t$

Using boundary conditions, ($x(0) = x_0$, $\dot{x}(0) = v_0$)

$C = x_0$,

$D = v_0 + \frac{1}{2}A$

so that $x = x_0\cos t + (v_0 + \frac{1}{2}A)\sin t - \frac{1}{2}At\cos t + \frac{1}{2}Bt\sin t$

Now, $x(t_f) = 0 = x_0\cos t_f + (v_0 + \frac{1}{2}A)\sin t_f - \frac{1}{2}At_f\cos t_f + \frac{1}{2}Bt_f\sin t_f$ (8)

$\dot{x}(t_f) = 0 = -x_0\sin t_f + (v_0 + \frac{1}{2}A)\cos t_f - \frac{1}{2}A\cos t_f + \frac{1}{2}At_f\sin t_f + \frac{1}{2}Bt_f\cos t_f + \frac{1}{2}B\sin t_f$ (9)

Eqns. (8) & (9) yield

$$\left.\begin{aligned}A &= -2\left[(x_0\cos t_f + v_0\sin t_f)\sin t_f + x_0 t_f\right]/(\sin^2 t_f - t_f^2)\\ B &= -2\left[(v_0\cos t_f - x_0\sin t_f)\sin t_f - v_0 t_f\right]/(\sin^2 t_f - t_f^2)\end{aligned}\right\} \quad (10)$$

Hence, $u = \dfrac{-2}{t_f^2 - \sin^2 t_f}\left[\sin(t_f - t)\sin t_f - t_f\sin t - \cos(t_f - t)\sin t_f + t_f\cos t\right]\begin{bmatrix}x_0\\ v_0\end{bmatrix}$

PROBLEM 5.4.1

$$\dot{x} = Fx + Gu \quad , (F, G) \text{ is a controllable pair.}$$

$$J = \min_{u(t)} \int_{t_0}^{t_f} \left(\frac{1}{2} \|x\|_A^2 + \frac{1}{2} \|u\|_B^2 \right) dt \qquad (1)$$

i) Upper bound on J :

Let $u^*(t)$ be a control which brings the system from $x(t_0)$ to the origin at time t_1, $t_0 < t_1 < t_f$. Controllability of (F, G) implies existence of such a control; moreover $u^*(t)$ is finite. Clearly, $x(t)$ remains finite under $u^*(t)$, since a finite control cannot produce unbounded states in a finite time. If we hold $u^*(t)$ at 0 for $t > t_1$ then the state $x(t) \equiv 0$ for $t \geq t_1$. This control $u^*(t)$ [with the associated state $x^*(t)$] which need not be optimal assures us an upper bound on J :

$$J \leq \int_{t_0}^{t_1} \left(\frac{1}{2} \|x^*\|_A^2 + \frac{1}{2} \|u^*\|_B^2 \right) dt \triangleq M < \infty \qquad (2)$$

(Note that $x^*(t) \equiv 0$, $u^*(t) \equiv 0$ for $t > t_1$)

Obviously M in (2) is not a function of t_f, so

$$\lim_{t_f \to \infty} J \leq M \qquad (3)$$

ii) We now show that J is always increasing as $t_f \to \infty$; and hence J is convergent.

Consider two possible terminal times t_{f_1}, t_{f_2} ($t_{f_1} \leq t_{f_2}$) and the corresponding optimal controls u_{01}, u_{02}. Note the following implicit facts in this notation:

a) $u_{02} \equiv u_{01}$ for $t \leq t_{f_1}$

b) The initial state for both the controls u_{01}, u_{02} is $x_0 \equiv x(t_0)$

Then,
$$J(x_0, t_0; t_{f_1}) = \int_{t_0}^{t_{f_1}} \left\{ \tfrac{1}{2} \| x_{01} \|_A^2 + \tfrac{1}{2} \| u_{01} \|_B^2 \right\} dt$$
$$= \int_{t_0}^{t_{f_1}} \left\{ \tfrac{1}{2} \| x_{02} \|_A^2 + \tfrac{1}{2} \| u_{02} \|_B^2 \right\} dt$$
$$\leq \int_{t_0}^{t_{f_2}} \left\{ \tfrac{1}{2} \| x_{02} \|_A^2 + \tfrac{1}{2} \| u_{02} \|_B^2 \right\} dt$$
$$= J(x_0, t_0; t_{f_2})$$

$$\Rightarrow \quad J(x_0, t_0; t_{f_1}) \leq J(x_0, t_0; t_{f_2}) \quad , \text{ for } t_{f_1} \leq t_{f_2} \qquad (4)$$

The equality in (4) holds only if $x_{02} \equiv 0$, $u_{02} \equiv 0$ for $t_{f_1} < t \leq t_{f_2}$; (since $A > 0$, $B \geq 0$) in this case the system remains at $x(t) \equiv 0$ for $t > t_{f_1}$. Thus, $J(x_0, t_0, t_f)$ is,

either a) always increasing as $t_f \to \infty$, but bounded from above ($J \leq M$)

or, b) converges in a finite time

In any case, $J(x_0, t_0, t_f)$ converges as $t_f \to \infty$,

$$J \longrightarrow J^* = \text{Const.}$$

(Note: $J = \frac{1}{2} x_0^T S(t_0) x_0 \longrightarrow J^* = \text{const} \Rightarrow S(t_0) \underset{t_f \to \infty}{\longrightarrow} S_0 = \text{const}$.)

Thus, (F, G) controllable $\Rightarrow$ steady state S_0 exists, and the linear feed back control law (refer eq. 5.2.27 in text) becomes a constant linear feed back control as $t_f \to \infty$.

With this control law $\left(u = -B^{-1} G^T S_0 x \right)$

we have, $\dot{x} = (F - GB^{-1}G^T S_0)x$, $x(t_0)$ given. (5)

(iii) Asymptotic stability of (5):

By Appendix B4 (ref. text p. 458) a sufficient condition for asymptotic stability (UAGS) is the existence of a Lyapunov function.

Let $V(x,t) = \frac{1}{2} \| x(t) \|^2_{S_0}$ (6)

a) Clearly $V(x,t) > 0$ if $x \neq 0$, since by construction $J = \frac{1}{2} x^T S_0 x \geq 0$, equality holding for $x \equiv 0$, $u \equiv 0$

b) $$\frac{dV(x,t)}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x}$$

$$= 0 + x^T S_0 (F - GB^{-1}G^T S_0) x$$

$$= \frac{1}{2} x^T [S_0 F + F^T S_0 - S_0 G B^{-1} G^T S_0 + A] x$$

$$- \frac{1}{2} x^T [A + S_0 G B^{-1} G^T S_0] x$$

Note that, A] $x^T S_0 F x = \frac{1}{2} x^T S_0 F x + \frac{1}{2} (x^T S_0 F x)^T$

$$= \frac{1}{2} x^T S_0 F x + \frac{1}{2} x^T F^T S_0 x$$

B] $-\dot{S}_0 = 0 = S_0 F + F^T S_0 - S_0 G B^{-1} G^T S_0 + A$.

A] and B] $\Rightarrow$ $\frac{dV(x,t)}{dt} = -\frac{1}{2} x^T [A + S_0 G B^{-1} G^T S_0] x$ (7)

But, $S_0 G B^{-1} G^T S_0 \geq 0$ and $A > 0$.

Hence $\frac{dV(x,t)}{dt} < 0$ whenever $x \neq 0$,

While $V(x,t) > 0$ if $x \neq 0$.

$\Longrightarrow$ $V(x,t)$ is a Lyapunov function, so (5) is asymptotically stable.

(Note: Other conditions in theorem B.6 on p. 458 are easily verified)

PROBLEM 5.4.3

Putting in the standar form, the system eqns become, $\dot{x} = Fx + Gu$, where

$$x = \begin{bmatrix} \alpha \\ q \end{bmatrix} , \quad F = \begin{bmatrix} -1/\tau & 1 \\ -\omega_0^2 & 0 \end{bmatrix} , \quad G = \begin{bmatrix} 0 \\ \omega_0^2 Q \end{bmatrix} , \quad u = \delta \quad . \qquad (1)$$

We wish to find steady state regulator gains C_1 and C_2 ($\delta = -C_1 \alpha - C_2 q$) to minimize,

$$J = \lim_{t_f \to \infty} \int_{t_0}^{t_f} \left\{ \tfrac{1}{2} \| x \|_A^2 + \tfrac{1}{2} \| u \|_B^2 \right\} dt \qquad (2)$$

where,

$$A = \begin{bmatrix} \frac{1}{\alpha_0^2} & 0 \\ 0 & \frac{1}{q_0^2} \end{bmatrix} , \quad B = \left[\frac{1}{u_0^2} \right] \equiv \frac{1}{\delta_0^2} \qquad (3)$$

The optimal steady state controller is,

$$u = -B^{-1} G^T S x \equiv -K x \quad \text{where } K = [C_1 \; C_2] , \qquad (4)$$

and S is the solution to the equation,

$$\dot{S} = 0 = -SF - F^T S - A + S G B^{-1} G^T S \qquad (5)$$

Equation is equivalent to 3 scalar equations:

$$0 = -2 S_{12} + \omega_0^4 \delta_0^2 Q^2 S_{22}^2 - \frac{1}{q_0^2} \qquad (6)$$

$$0 = -S_{11} + \frac{S_{12}}{\tau} + \frac{S_{22}}{\omega_0^2} + \omega_0^4 \delta_0^2 Q^2 S_{11} S_{22} \qquad (7)$$

$$0 = \frac{2 S_{11}}{\tau} + 2 S_{12} \omega_0^2 + \omega_0^4 \delta_0^2 Q^2 S_{12}^2 - \frac{1}{\alpha_0^2} \qquad (8)$$

Eqs. (6), (7), (8) may be solved as they are, but a lot of simplicity results if we let $\frac{1}{\tau} \to 0$. This is physically reasonable, since the time over which a significant change in attack angle

α occurs must be very large compared with the time required to move the control surfaces (the control would be ineffective otherwise).

Eq. (8) then is a quadratic in S_{12} and

$$\Rightarrow \quad S_{12} = \frac{1}{\delta_0^2 \omega_0^2 Q^2}\left[-1 + \sqrt{1+\left(\frac{Q\delta_0}{\alpha_0}\right)^2}\right] \tag{9}$$

$$\text{Eq. (6)} \Rightarrow \quad S_{22} = \frac{1}{\delta_0^2 \omega_0^2 Q^2}\left[\frac{1}{q_0^2} + 2S_{12}\right]^{1/2} \tag{10}$$

Since,

$$\begin{aligned} \delta &= -\bar{B}^{-1} G^T S x \\ &= -\delta_0^2 \omega_0^2 Q \begin{bmatrix} S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} \end{aligned} \tag{11}$$

we will not need S_{11}.

$$\text{Eq. (11)} \Rightarrow \quad C_1 = \delta_0^2 \omega_0^2 Q S_{12}$$

$$= \frac{1}{Q}\left[-1 + \sqrt{1+\left(\frac{Q\delta_0}{\alpha_0}\right)^2}\right], \tag{12}$$

$$C_2 = \delta_0^2 \omega_0^2 Q S_{22}$$

$$= \frac{1}{Q}\left[\frac{1}{q_0^2} + \frac{2}{\delta_0^2 \omega_0^2 Q^2}\left(-1 + \sqrt{1+\left(\frac{Q\delta_0}{\alpha_0}\right)^2}\right)\right]^{1/2} \tag{13}$$

Under the assumption $\frac{1}{\tau} \to 0$, the controlled system can be written as follows:

$$\dot{\alpha} = q$$

$$\ddot{\alpha} = \dot{q} = -\omega_0^2 (\alpha + Q C_1 \alpha + Q C_2 \dot{\alpha})$$

$$\Rightarrow \quad 0 = \ddot{\alpha} + \omega_0^2 Q C_2 \dot{\alpha} + \omega_0^2 (1 + Q C_1)\alpha \tag{14}$$

Comparing eq. (14) with the standard form for a second order system, i.e.

$$\ddot{\alpha} + 2\zeta\omega\dot{\alpha} + \omega^2\alpha = 0 \quad , \tag{15}$$

we see that the natural frequency in the present case is,

$$\omega = \omega_0 \left[1 + \left(\frac{Q\delta_0}{\alpha_0}\right)^2\right]^{1/4} \tag{16}$$

and, the damping constant ζ is,

$$\zeta = \frac{\omega_0}{2} \sqrt{\frac{\frac{1}{q_0^2} + \frac{2}{\delta_0^2\omega_0^2Q^2}\left(-1 + \sqrt{1+\left(\frac{Q\delta_0}{\alpha_0}\right)^2}\right)}{\sqrt{1 + \left(\frac{Q\delta_0}{\alpha_0}\right)^2}}} \tag{17}$$

Recognizing two dimensionless groupings, ie

$$x \equiv \frac{Q\delta_0\omega_0}{q_0} \quad \text{and}$$

$$y \equiv \frac{Q\delta_0}{\alpha_0}$$

we can rewrite (17) as

$$\zeta = \frac{\omega_0}{2q_0} \sqrt{\frac{1 + \frac{2}{x^2}\left(-1 + \sqrt{1+y^2}\right)}{\sqrt{1+y^2}}} \quad . \tag{18}$$

Noting that $\frac{\omega_0}{q_0}$ is just a constant (dimensionless) it is now easy to obtain contours of constant damping ratio ζ on a plot of

$$y \equiv \frac{Q\delta_0}{\alpha_0} \quad \text{vs.} \quad x \equiv \frac{Q\delta_0\omega_0}{q_0}.$$

(Reviewing Sec. 4-3 would be of much help for this problem.)

PROBLEM 5.4.4.

We make the following identifications:

$$x = \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}, \quad F = \begin{bmatrix} -\frac{1}{\tau} & 1 & 0 \\ -\omega_0^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \omega_0^2 Q \\ 0 \end{bmatrix}, \quad u = \delta$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \frac{1}{\gamma_0^2}, \quad B = \frac{1}{\delta_0^2}, \quad S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

Substituting into $\dot{S} = 0 = -SF - F^T S - A + S G B^{-1} G^T S$,

we get 6 scalar equations:

$$\left.\begin{aligned}
\frac{2}{\tau} S_{11} + 2\omega_0^2 S_{22} + K S_{12}^2 &= \frac{1}{\gamma_0^2} \\
-S_{11} + \frac{1}{\tau} S_{12} - S_{13} + \omega_0^2 S_{22} + K S_{12} S_{22} &= 0 \\
\frac{1}{\tau} S_{13} + \omega_0^2 S_{23} + K S_{12} S_{23} &= -\frac{1}{\gamma_0^2} \\
-2 S_{12} - 2 S_{23} + K S_{22}^2 &= 0 \\
-S_{13} - S_{33} + K S_{22} S_{23} &= 0 \\
K S_{23}^2 &= \frac{1}{\gamma_0^2}
\end{aligned}\right\} \quad K = \omega_0^4 Q^2 \delta_0^2$$

Then, $\delta = -B^{-1} G^T S x = -\delta_0^2 \omega_0^2 Q \begin{bmatrix} S_{12} & S_{22} & S_{23} \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$

$$\Rightarrow \quad C_j = \delta_0^2 \omega_0^2 Q S_{2j} \quad ; \quad j = 1, 2, 3$$

S_{ij} must be such that $S > 0$. (Once again the solution is easy if we take $\frac{1}{\tau} \to 0$)

PROBLEM 5.4.5

As in previous problem we make the identifications:

$$x^T = [\alpha \quad q \quad \theta \quad h], \quad B = \frac{1}{\delta_0^2}, \quad u = \delta$$

$$F = \begin{bmatrix} -\frac{1}{\tau} & 1 & 0 & 0 \\ -\omega_0^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -V & 0 & V & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \omega_0^2 Q \\ 0 \\ 0 \end{bmatrix}, \quad A_{ij} = 0 \text{ except for } A_{44} = \frac{1}{h_0^2}.$$

$\dot{S} = 0$ reduces to 10 scalar equations:

$$0 = \frac{2}{\tau} S_{11} + 2\omega_0^2 S_{12} + 2V S_{14} + K S_{12}^2$$

$$0 = -S_{11} - S_{13} + \frac{1}{\tau} S_{12} + \omega_0^2 S_{22} + V S_{24} + K S_{12} S_{22}$$

$$0 = -V S_{14} + \frac{1}{\tau} S_{13} + \omega_0^2 S_{23} + V S_{34} + K S_{12} S_{23}$$

$$0 = \frac{1}{\tau} S_{14} + \omega_0^2 S_{24} + V S_{44} + K S_{12} S_{24}$$

$$0 = -2S_{12} - 2S_{23} + K S_{22}^2$$

$$0 = -V S_{24} - S_{13} - S_{33} + K S_{22} S_{23}$$

$$0 = -S_{14} - S_{34} + K S_{22} S_{24}$$

$$0 = -2V S_{34} + K S_{23}^2$$

$$0 = -V S_{44} + K S_{23} S_{24}$$

$$\frac{1}{h_0^2} = K S_{24}^2 \qquad \text{where} \quad K = \omega_0^4 Q^2 \delta_0^2$$

As before, $c_j = \delta_0^2 \omega_0^2 Q S_{2j}$; $j = 1, 2, 3, 4$.

We can solve the 10 eqns. above for S_{2j}.

PROBLEM 5.4.6 (Solution is in text.)

PROBLEM 5.4.7

$$\dot{x} = Fx + Gu \quad , \quad x \Rightarrow (n\times 1) \, , \quad u \Rightarrow (m\times 1) \tag{1}$$

$$y = Mx \quad , \quad y \Rightarrow (p\times 1) \tag{2}$$

$$\min_u J = \frac{1}{2}\int_{t_0}^{t_f} \{(y-Mx)^T A (y-Mx) + u^T B u\}\, dt \tag{3}$$

$$H = \frac{1}{2}\left[y^T A y - x^T M^T A y - y^T A M x + x^T M^T A M x + u^T B u \right] + \lambda^T (Fx + Gu) \tag{4}$$

$$0 = H_u \Rightarrow \quad u = -B^{-1} G^T \lambda \tag{5}$$

$$\dot{\lambda} = -H_x^T = M^T A y - M^T A M x - F^T \lambda \, , \quad \lambda(t_f) = 0 \tag{6}$$

Using sweep method let,

$$\lambda(t) = S(t)\, x(t) + g(t) \tag{7}$$

Differentiating eq.(7) and using eqs (6), (1) & (5),

$$M^T A y - M^T A M x - F^T (Sx+g) = S\dot{x} + \dot{S}x + \dot{g}$$

$$= S\left[Fx - GB^{-1}G^T (Sx+g)\right] + \dot{S}x + \dot{g}$$

Collecting coefficients of like terms,

$$\dot{S} = -SF - F^T S + SGB^{-1}G^T S - M^T A M \, , \quad S(t_f) = 0$$

$$\dot{g} = -(F^T - SGB^{-1}G^T) g + M^T A y \, , \quad g(t_f) = 0$$

$$u = -B^{-1}G^T (Sx+g) = -Cx + w$$

$$\Rightarrow \quad C = B^{-1}G^T S \, , \quad w = -B^{-1}G^T g$$

PROBLEM 5.4.8

$$\dot{x} = F(t)\, x + G(t)\, u + C(t) \, , \quad x(t_0) \text{ given} \tag{i}$$

$$J = \frac{1}{2}\left(x^T S_f x\right)_{t_f} + \frac{1}{2}\int_{t_0}^{t_f} (x^T A x + u^T B u)\, dt \qquad (2)$$

$$H = \frac{1}{2}(x^T A x + u^T B u) + \lambda^T (Fx + Gu + C) \qquad (3)$$

$$\Rightarrow \begin{cases} \dot{\lambda} = -Ax - F^T\lambda \\ \lambda(t_f) = S_f\, x(t_f) \end{cases} \qquad (4)$$

$$u = -B^{-1}G^T\lambda \qquad (5)$$

Using the sweep method to solve the TPBVP [eqs. (1) & (4)] let, $\lambda = Sx + k$

Then proceeding as in previous problem,

$$\dot{S} = -SF - F^TS - A + SGB^{-1}G^TS \quad , \quad S(t_f) = S_f$$

$$\dot{k} = (GB^{-1}G^T - F^T)k - SC \quad , \quad k(t_f) = 0$$

$$\Rightarrow u = -B^{-1}G^T(Sx + k).$$

PROBLEM 5.4.9

$$\dot{x} = \underset{(n\times 1)}{F(t)\,x} + w(t) \quad , \quad A\,x(t_0) = \underset{[(n-k)\times 1]}{a} \quad , \quad Bx(t_f) = \underset{(k\times 1)}{b} \qquad (1)$$

$F(t)$, $w(t)$, A, B, a, b are given.

Note: This is a generalization of the TPBVP on p. 149 of text, eqs. 5.2.7, 5.2.8. One could look at the x in the present problem as $\begin{bmatrix} x \\ \lambda \end{bmatrix}$ of the TPBVP in the text.

FIRST METHOD OF SOLUTION :

In an attempt to put eq.(1) in the framework of the eqns 5.2.7 and 5.2.8, one needs to partition F(t) into 4 blocks. As suggested in the solution to the problem, we look at the "similarity transformation",

$$\left[\begin{array}{c} A \\ \hline B \end{array}\right] F(t) \left[\begin{array}{c} A \\ \hline B \end{array}\right]^{-1} = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right] \tag{2}$$

Note that, $\left[\begin{array}{c} A \\ \hline B \end{array}\right]$ is square and must be invertible for a complete specification of the problem (n boundary conditions or n independent linear combinations of x should be specified)

a) For a forward sweep, assume

$$A\,x(t) = S(t)\,Bx(t) + m(t) \tag{3}$$

$$\Rightarrow \quad A\,\dot{x} = S\,B\,\dot{x} + \dot{S}Bx + \dot{m}$$

$$\text{or,} \quad A(Fx+w) = SB(Fx+w) + \dot{S}Bx + \dot{m} \tag{4}$$

But eq (2) may be rewritten,

$$\left[\begin{array}{c} AF \\ \hline BF \end{array}\right] = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right] \left[\begin{array}{c} A \\ \hline B \end{array}\right] = \left[\begin{array}{c} C_1A + C_2B \\ \hline C_3A + C_4B \end{array}\right] \tag{5}$$

$$\Rightarrow \begin{cases} AF = C_1A + C_2B & (6) \\ BF = C_3A + C_4B & (7) \end{cases}$$

Hence eq. (4) becomes,

$$(C_1A + C_2B)x + Aw = S(C_3A + C_4B)x + SBw + \dot{S}Bx + \dot{m}$$

But, $Ax = SBx + m$, so

$$C_1(SBx + m) + C_2Bx + Aw = SC_3(SBx + m) + SC_4Bx + SBw + \dot{S}Bx + \dot{m}$$

or, $$(\dot{S} - C_1S + SC_4 + SC_3S - C_2)Bx + \dot{m} + (C_1 - SC_3)m + (A - SB)w = 0 \quad (8)$$

Also, $$Ax(t_0) = S(t_0)Bx(t_0) + m(t_0) = a$$

$$\Rightarrow m(t_0) = a, \quad S(t_0) = 0 \quad (9)$$

Eqns. (8) & (9) yield,

$$\dot{S} = C_1S - SC_4 - SC_3S + C_2, \quad S(t_0) = 0 \quad (10)$$

$$\dot{m} = (C_1 - SC_3)m + (A - SB)w, \quad m(t_0) = a \quad (11)$$

Integrating eqns. (10) & (11) forward generates a family of solutions that satisfies that satisfies the initial conditions.

At $t = t_f$, we have,

$$Ax(t_f) = S(t_f)b + m(t_f), \quad Bx(t_f) = b \quad (12)$$

These are n eqns. for $x(t_f)$ and can be solved for $x(t_f)$, and then eq. (1) is integrated backwards

Note that since $\left[\frac{A}{B}\right]$ is invertible,

$$x(t_f) = \left[\frac{A}{B}\right]^{-1}\left\{\left[\frac{S(t_f)}{I_k}\right]b + \left[\frac{m(t_f)}{0}\right]\right\}$$

b) Backward sweep using $Bx(t) = QAx(t) + n(t)$ could be carried out as in (a) above.

SECOND METHOD OF SOLUTION :

Since $S(t)$ occurs always in conjunction with B as $S(t)B$ in eq. (4), a more direct 'sweep' would be to recognize that $S(t)B$ could be swept rather than $S(t)$. Equation (4) yields directly,

$$\dot{S}B \equiv \frac{d}{dt}(SB) = (A - SB)F \,, \quad S(t_0)B = 0 \tag{13}$$

$$\dot{m} = (A - SB)w \,, \quad m(t_0) = a \tag{14}$$

Integrating eqns. (13) and (14) forward generates a family of solutions that satisfies the initial conditions.

At $t = t_f$, we have,

$$Ax(t_f) = S(t_f)Bx(t_f) + m(t_f) \,, \quad Bx(t_f) = b$$

$$\Rightarrow \quad x(t_f) = \left[\frac{A}{B}\right]^{-1} \left\{ \left[\frac{S(t_f)}{I_k}\right] b + \left[\frac{m(t_f)}{0}\right] \right\} \tag{15}$$

Having got $x(t_f)$ from eq. (15), eq. (1) may be integrated backwards to yield,

$$x(t) = \phi(t, t_f)\, x(t_f) + \int_{t_f}^{t} \phi(t, \tau)\, w(\tau)\, d\tau \tag{16}$$